Enumeration of Walks on Generalized Differential Posets

by

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Abstract

Many enumerative problems can be restated as questions about walks on the cover relations of a partially ordered set. Differential posets are a class of posets in which one can solve the walk-enumeration problem explicitly. The technique relies on the use of linear operators U and D, defined on the formal span of a poset $\mathcal{P}$, which satisfy the commutation relation $DU - UD = rI$ for some constant r. By making weaker assumptions about U and D, the linear algebraic techniques used to enumerate walks on differential posets can be applied in a more general context. This thesis is a uniform presentation of the work done in this area, mainly due to Stanley and Fomin. It also includes new proofs or minor generalizations of some results, a new example of a generalized differential poset, and a proof of a "folklore theorem" about taking quotients of generalized differential posets modulo a group action.

Acknowledgements

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Chapter 1

Introduction

This thesis deals with the problem of enumerating walks on differential posets, and on posets which are generalizations of differential posets. It is a uniform presentation of the main results related to this problem, along with new proofs or minor generalizations of some results. Results which have been omitted in order to keep this thesis at a reasonable length are summarized at the end of Chapter 3.

Partially ordered sets are a useful tool for answering problems about enumerative combinatorics. Many combinatorial problems can be restated as enumerative questions about posets. An important general class of such questions is the counting of walks on the cover relations of the poset. Differential posets are a class of partially ordered sets which have enough structure to allow one to solve the walk-enumeration problem explicitly. This thesis deals with the linear algebraic techniques used to approach the walk-enumeration problem on differential posets and generalizations thereof.

The theory of differential posets is presented in Chapter 2. Differential posets were first introduced by Stanley [17] in a paper where he develops the linear algebraic theory used to enumerate walks on differential posets. (There is another approach to this problem using symmetric functions, due to Gessel [5], though this technique is not explored in this thesis.) My contributions to the theory expounded in this thesis include an expanded exposition of some foundational issues in Sections 2.1 and 2.2, and a new proof of Theorem 2.42, in which a partial differential equation is solved using the method of characteristics.

Chapter 3 deals with generalized differential posets. These structures are algebraically equivalent to the oriented graded graphs of Fomin [3], and many of the results presented in Chapter 3 were first proven in the context of oriented graded graphs. A special case of generalized differential posets, called sequentially differential posets, was also studied by Stanley [18]. Contributions made in this thesis include a mild generalization of Fomin's results on the spectral theory of oriented graded graphs, and Theorem 3.12, which allows one to solve the walk-enumeration problem on a set of shapes of walks which is more general than that solved in [3].

Chapter 4 develops the theory of taking quotients of generalized differential posets with respect to rank-compatible equivalence relations, and in particular, with respect to the orbits of a group of automorphisms of the poset. While the results in this chapter do not seem to appear in the literature, they resemble results on PECK posets due to Stanley [16]. It is generally known that the proofs of these results on PECK posets can also be applied to generalized differential posets; in this thesis, we give a precise statement and proof of this "folklore theorem."

Throughout this thesis, a number of examples are presented which connect the walk-enumeration problem on posets to other combinatorial problems. Enumeration of walks on the classical Young's lattice, discussed in the present chapter, allows one to enumerate standard Young tableaux. A partially ordered set of rooted trees, introduced by Hoffman [8], is presented in Section 3.1.1, and allows one to deal with questions related to the growth of rooted trees. The poset of spanning subgraphs of K_n (the complete graph on n vertices) is presented in Chapter 4, and the spectral theory of generalized differential posets from Chapter 3 is used to solve a special case of the edge-reconstruction conjecture of Harary [7]. This case was first solved by Lovász [9], and the proof given in this thesis is very similar to Stanley's proof using the theory of PECK posets [16]. A new example of a generalized differential poset, the induced subgraph order on unlabelled graphs, is introduced in Chapter 3.

1.1 Motivating Example: Young's Lattice

In this section, Young's Lattice, the prototypical differential poset, is introduced as a motivating example. Some details in the proofs are omitted, since they will be covered in later chapters. The purpose of this section is

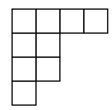


Figure 1.1: Ferrers diagram of the partition (4221)

to give an idea of the flavour of the rest of the thesis. In order to discuss Young's Lattice, a number of definitions will be needed. The following may be found in a variety of sources, such as Macdonald's book on symmetric functions [10]. Let n be a non-negative integer. A **partition of** n is a weakly decreasing sequence of integers

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$$

such that $\sum_{i=1}^{k} \lambda_i = n$. Each λ_i is called a **part** of the partition. (It is often helpful to think of a partition as an infinite sequence, with $\lambda_i = 0$ for $i > k$.) If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is such a sequence, we write $\lambda \vdash n$. For notational convenience, a partition with k parts is often written with neither trailing zeroes nor commas, as $\lambda = (\lambda_1 \lambda_2 \ldots \lambda_k)$. The unique partition of 0 consisting of no parts is denoted by $\hat{0}$.

A partition may be represented by its **Ferrers diagram** F_λ, which consists of k rows of boxes, with λ_i boxes in each row, flush with the left margin. This may be defined more formally as

$$F_\lambda := \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq k, 1 \leq j \leq \lambda_i\}.$$

As an example, the Ferrers diagram of the partition $(4221) \vdash 9$ is shown in Figure 1.1.

Given a partition $\lambda \vdash n$, we can form a **standard Young tableau** of shape λ (hereafter referred to simply as a "tableau") by assigning the integers $\{1, 2, \ldots n\}$ to the boxes of F_λ so that the integers are increasing from left to right along rows, and from top to bottom down columns. More formally, it is a bijection $\phi : F_\lambda \to \{1, 2, \ldots n\}$ such that $\phi(i+1, j) > \phi(i, j)$ and $\phi(i, j+1) > \phi(i, j)$. Two tableaux of shape (4221) are shown in Figure 1.2. Denote by f_λ the number of tableaux of shape λ.

A similar object, which is used in later examples, is a **skew tableau**. Given two partitions λ and μ, with $F_\mu \subset F_\lambda$, define the diagram $F_{\lambda\setminus\mu}$ to be $F_\lambda \setminus F_\mu$ — that is, $F_{\lambda\setminus\mu}$ is the Ferrers diagram for λ with the boxes

1	3	4	5
2	6		
7	8		
9			

1	3	6	9
2	5		
4	8		
7			

Figure 1.2: Two of the standard Young tableaux of shape (4221)

		4	5
	1		
2	3		
6			

Figure 1.3: Skew tableau of shape $(4221) \setminus (21)$

corresponding to μ removed. As before, we define a skew tableau of shape $\lambda \setminus \mu$ to be a bijection $\phi : F_{\lambda\setminus\mu} \to \{1, 2, \ldots, |F_{\lambda\setminus\mu}|\}$ such that $\phi(i+1, j) > \phi(i, j)$ and $\phi(i, j+1) > \phi(i, j)$, which is represented diagrammatically as an assignment of integers to the boxes of $F_{\lambda\setminus\mu}$. An example of a skew tableau is shown in Figure 1.3.

A well-known equation relating the numbers of tableaux of a given shape is

$$\sum_{\lambda \vdash n} (f_\lambda)^2 = n!. \tag{1.1}$$

There are several proofs of this result. It can be proven in a purely combinatorial manner by means of the Robinson-Schensted correspondence [14], which is a bijection between permutations on n symbols and pairs of standard Young tableaux of the same shape $\lambda \vdash n$. It can also be proven using the representation theory of the symmetric group, where it is recognized as the formula for the sum of the squares of the degrees of the irreducible representations of the symmetric group. (See, for example, Theorem 2.6.5 of [13].) This chapter will deal with a third, algebraic-combinatorial proof, which is a special case of the results of Stanley in [17], and which is also discussed in [13].

The definitions pertaining to partially ordered sets which are needed to understand this thesis may be found in Appendix A. The poset-theoretic proof of (1.1) relies on regarding the set $\mathcal{Y}$ of all partitions as a partially ordered set, with the order given by $\lambda \leq \mu$ if and only if $F_\lambda \subseteq F_\mu$. Geometrically, this means that if F_λ is placed on top of F_μ such that the upper left corners of the diagrams coincide, then the diagram F_λ is contained entirely within the diagram F_μ. It is easy to verify that $\mathcal{Y}$ is a lattice (in fact, it is just the lattice $J(\mathbb{N}^2)$ of downward-closed subsets of $\mathbb{N}^2$), called **Young's lattice**.

In Young's lattice, a useful characterization of the cover relation is that $\lambda \lessdot \mu$ if and only if $F_\mu \setminus F_\lambda$ consists of exactly one element. Young's lattice may be represented as an infinite graph whose vertices are partitions, and two vertices λ and μ are adjacent if and only if $\lambda \lessdot \mu$ or $\mu \lessdot \lambda$ in the partial order on $\mathcal{Y}$. This graph is called the **Hasse diagram** of $\mathcal{Y}$, and is shown in Figure 1.4. Walks on the Hasse diagram of a poset are called **Hasse walks**.

For any element $\lambda \in \mathcal{Y}$, let $\mathcal{U}_\lambda$ be the set of all partitions which cover λ, that is,

$$\mathcal{U}_\lambda = \{\mu \in \mathbb{Y} : \lambda \lessdot \mu\}.$$

Similarly, let $\mathcal{D}_\lambda$ be the set of all partitions covered by λ, that is,

$$\mathcal{D}_\lambda = \{\mu \in \mathbb{Y} : \mu \lessdot \lambda\}.$$

Two combinatorial facts about these sets are used in the linear algebraic study of Young's Lattice:

$$\text{if } \lambda \neq \mu, \text{ then } |\mathcal{U}_\lambda \cap \mathcal{U}_\mu| = |\mathcal{D}_\lambda \cap \mathcal{D}_\mu| \tag{1.2}$$

and

$$|\mathcal{U}_\lambda| = |\mathcal{D}_\lambda| + 1. \tag{1.3}$$

Equation (1.2) may be proven by first showing that $\mathcal{U}_\lambda \cap \mathcal{U}_\mu$ is non-empty if and only if $\mathcal{D}_\lambda \cap \mathcal{D}_\mu$ is non-empty. Suppose $\pi \in \mathcal{U}_\lambda \cap \mathcal{U}_\mu$. Then $F_\pi \setminus F_\lambda = \{(i_1, j_1)\}$ and $F_\pi \setminus F_\mu = \{(i_2, j_2)\}$. Note that (i_1, j_1) and (i_2, j_2) are distinct, for otherwise, $\lambda = \mu$. But $F_\pi \setminus \{(i_1, j_1), (i_2, j_2)\}$ is in both $\mathcal{D}_\lambda$ and $\mathcal{D}_\mu$, so the intersection of these sets is non-empty. This argument is symmetric, so the claim

$$\mathcal{U}_\lambda \cap \mathcal{U}_\mu \neq \emptyset \Leftrightarrow \mathcal{D}_\lambda \cap \mathcal{D}_\mu \neq \emptyset$$

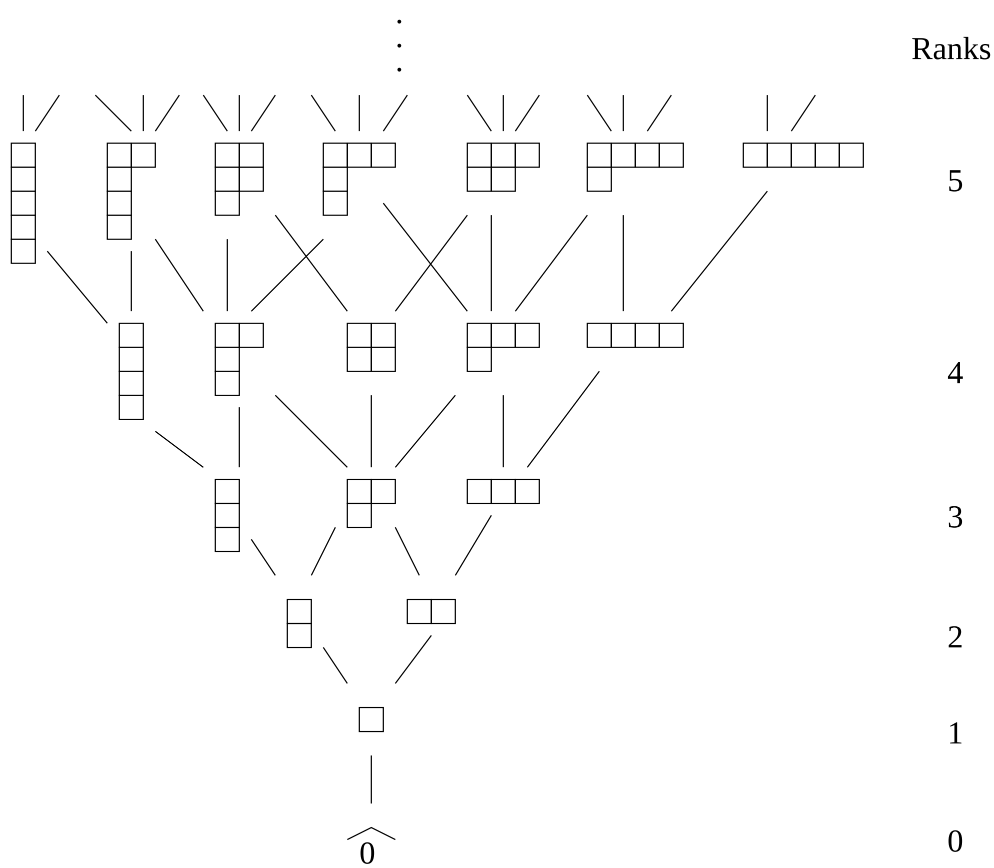


Figure 1.4: Young's Lattice

is in fact an "if and only if" statement. In the case where these intersections are non-empty, we now show that they must contain a single element. Suppose $\pi, \pi' \in \mathcal{U}_\lambda \cap \mathcal{U}_\mu$. Then

$$F_\pi = F_\lambda \cup \{(i_1, j_1)\} = F_\mu \cup \{(i_2, j_2)\} \tag{1.4}$$

and

$$F_{\pi'} = F_\lambda \cup \{(i'_1, j'_1)\} = F_\mu \cup \{(i'_2, j'_2)\}, \tag{1.5}$$

where the unions are disjoint, $(i_1, j_1) \neq (i_2, j_2)$ and $(i'_1, j'_1) \neq (i'_2, j'_2)$. If $(i_2, j_2) = (i'_2, j'_2)$, we are done, so assume $(i_2, j_2) \neq (i'_2, j'_2)$. By equation (1.4), $(i_2, j_2) \in F_\lambda$. But then equation (1.5) implies that $(i_2, j_2) \in F_\mu$, contradicting the fact that $F_\pi \setminus F_\mu = (i_2, j_2)$. Thus $F_\pi = F_{\pi'}$, hence $\pi = \pi'$, and $|\mathcal{U}_\lambda \cap \mathcal{U}_\mu| = 1$. The argument for $\mathcal{D}$ is similar, so equation (1.2) is proven.

Equation (1.3) may be proven by first observing that $|\mathcal{D}_\lambda|$ is the number of boxes that, when removed from F_λ, leave the Ferrers diagram of a partition. Note that this number is equal to the number of parts λ_i such that $\lambda_i > \lambda_{i+1}$. (In this case, by removing the rightmost box from row i, we still have a Ferrers diagram of a partition.) On the other hand, whenever $\lambda_i > \lambda_{i+1}$, a box may be added to the right end of row $i+1$ to obtain a Ferrers diagram of a partition. But a box may also be added to the right end of the first row. Since this exhausts all possible positions where a single box may be added to F_λ, and the number of such positions is equal to $|\mathcal{U}_\lambda|$, equation (1.3) follows.

The problem of counting tableaux of shape λ is the same as the problem of counting walks on the Hasse diagram of Young's Lattice of the form

$$\hat{0} \lessdot \lambda_1 \lessdot \lambda_2 \lessdot ... \lessdot \lambda_n = \lambda.$$

There is a bijection between standard Young tableaux of shape λ and paths of this form. Given a tableau T, identify it with a walk where λ_i is the shape of the tableau we obtain from T by deleting all boxes except those containing the integers $1, 2, \ldots, i$. (Another way of looking at this bijection is that $F_{\lambda_i} \setminus F_{\lambda_{i-1}}$ is the box of T containing i.) An example of this bijection is shown in Figure 1.5.

Note that this bijection may be modified to deal with skew tableaux. In this case, the number of skew-tableau of shape $\lambda \setminus \mu$ is equal to the number of Hasse walks of the form

$$\mu \lessdot \lambda_1 \lessdot \ldots \lessdot \lambda_k = \lambda.$$

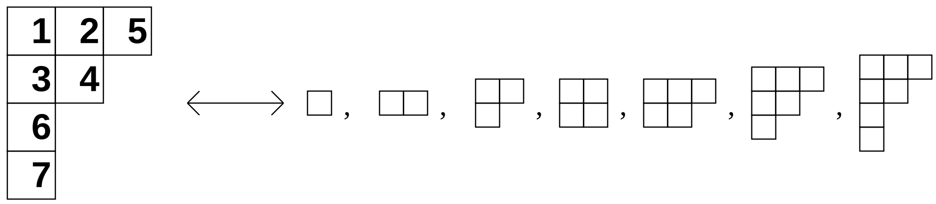


Figure 1.5: Example of bijection between standard Young tableaux and increasing paths in $\mathcal{Y}$

1.1.1 The Vector Space of Partitions

The linear algebraic proof of equation (1.1) uses the vector space $\mathbb{Q}\mathcal{Y}$ of formal, rational linear combinations of elements of $\mathcal{Y}$. Define linear transformations U and D on this vector space by

$$U(x) = \sum_{x \lessdot y} y$$

and

$$D(x) = \sum_{y \lessdot x} y,$$

extended linearly to all of $\mathbb{Q}\mathcal{Y}$. A combinatorial way of viewing these transformations is as follows. Let $a_n(x, y)$ denote the number of Hasse walks of the form

$$x \lessdot z_1 \lessdot \cdots \lessdot z_{n-1} \lessdot y.$$

Then

$$U(x) = \sum_{y \in \mathcal{Y}} a_1(x, y) y.$$

Proceeding by induction, it is not difficult to see that $U^n(x) = \sum_{y \in \mathcal{Y}} a_n(x, y) y$. (This is a special case of a result developed in more detail in Section 2.2.) From the discussion about the bijection between tableaux and walks on Young's lattice, we can see that

$$U^n \hat{0} = \sum_{\lambda \vdash n} f_\lambda \lambda.$$

By a similar argument,

$$D^n \lambda = f_\lambda \hat{0}$$

for $\lambda \vdash n$. Combining these, we obtain

$$D^n U^n \hat{0} = \sum_{\lambda \vdash n} (f_\lambda)^2 \hat{0}.$$

So, it suffices to compute $D^n U^n \hat{0}$ and examine the coefficient of $\hat{0}$.

Using (1.2) and (1.3), we obtain

$$DU - UD = I. \tag{1.6}$$

This can be proven by brute force computation of $DU - UD$; the details may be found in Section 2.1, where this appears as a special case of a more general result.

Applying (1.6), we obtain

$$DU^n = UDU^{n-1} + U^{n-1}$$

so, proceeding inductively,

$$DU^n = U^n D + nU^{n-1}. \tag{1.7}$$

Equation (1.7) may now be applied repeatedly to $D^n U^n \hat{0}$ to obtain

$$\begin{aligned}
\left(\sum_{\lambda \vdash n} (f_\lambda)^2 \right) \hat{0} &= D^n U^n \hat{0} \\
&= D^{n-1}(U^n D + nU^{n-1})\hat{0} \\
&= nD^{n-1}U^{n-1}\hat{0} \text{ (since } D\hat{0} = 0) \\
&\vdots \\
&= n!\hat{0}.
\end{aligned}$$

Equation (1.1) follows immediately by comparing coefficients of $\hat{0}$.

Chapter 2

Differential Posets

The example of Young's lattice in Chapter 1 demonstrates the usefulness of using the linear operators U and D to solve the problem of enumerating Hasse walks. More generally, our strategy for enumerating walks is as follows. Let $x_1 x_2 \ldots x_n$ be a walk on a poset $\mathcal{P}$, that is, either $x_i \lessdot x_{i+1}$ or $x_{i+1} \lessdot x_i$, for $1 \leq i \leq n-1$. Any such walk may be described as a sequence of steps, either up or down, in the partially ordered set. Using "U" to represent an up-step and "D" to represent a down-step, write the steps *from right to left* to obtain a monomial W encoding the shape of the walk. (The word is written in this way so that the order of steps in the walk of shape W corresponds to the composition of functions U and D.) More formally, W is defined by $W = S_{n-1} \ldots S_2 S_1$ where $S_i = U$ if $x_i \lessdot x_{i+1}$ and $S_i = D$ if $x_{i+1} \lessdot x_i$. The **length** of W, denoted $|W|$, is the total number of U's and D's in W. The **displacement** of W, denoted $\ell(W)$, is the number of U's in W minus the number of D's. For example, the walk (3)(31)(21)(11)(111)(211) on Young's lattice has shape $UUDDU$, length 5 and displacement 1. (See Figure 2.1.) Conversely, any monomial $W \in \{U, D\}^*$ describes a possible shape of a walk. By applying W to elements of the vector space spanned by the poset, we can obtain enumerative results pertaining to the number of Hasse walks of shape W.

In Section 2.1, a discussion of the algebra related to the operators U and D will lead to a definition of a differential poset, which was first defined by Stanley [17]. In Section 2.2, the connection between linear algebra and the walk-enumeration problem is made more precise. The remainder of this chapter describes further development of the algebra of differential posets

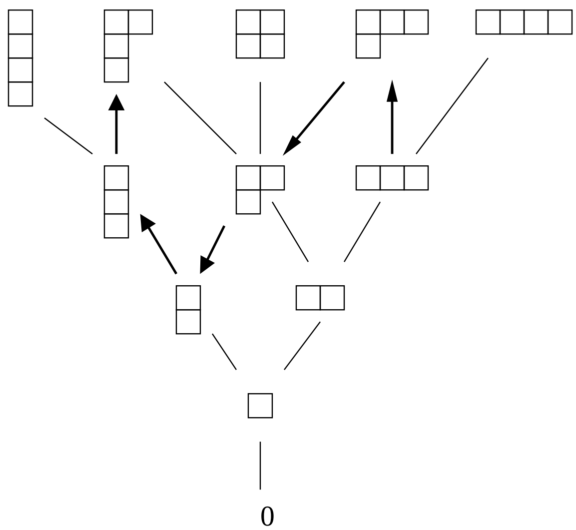


Figure 2.1: Example of a walk of shape $UUDDU$ in Young's Lattice

and its use to solve walk-enumeration problems. This is also due to Stanley.

2.1 Definition and Basic Results

This discussion takes place in the context of a poset $\mathcal{P}$ which

1. is locally finite,
2. is graded, with grading ρ,
3. has finitely many elements of each rank,
4. and has a least element $\hat{0}$.

It is clear, for example, that Young's Lattice satisfies all these conditions — the empty partition is the least element $\hat{0}$ and the rank function is $\rho(\lambda) = n$, where $\lambda \vdash n$. Our objective is to determine conditions under which the operators U and D satisfy $DU - UD = rI$, thereby generalizing a useful property of Young's Lattice.

Let $\mathbb{K}$ be a field of characteristic 0. Often, $\mathbb{K}$ will be the field $\mathbb{Q}((t))$ of formal Laurent series, though sometimes it is useful to consider fields with more indeterminates. As in Chapter 1, we would like to consider the vector space of formal linear combinations of elements of $\mathcal{P}$. However, we need to be able to deal with infinite linear combinations of elements of $\mathcal{P}$, so simply working with $\text{span}_{\mathbb{K}}\mathcal{P}$ is not sufficient. Some care must be taken to ensure that infinite linear combinations, and the operations on them, are well-defined. Thus, we work instead with the vector space $\mathbb{K}^{\mathcal{P}}$ of functions from $\mathcal{P}$ to $\mathbb{K}$. For convergence considerations, we use the discrete topology on $\mathbb{K}$, and extend it to $\mathbb{K}^{\mathcal{P}}$ as the product topology. Thus, informally, sequences of elements of $\mathbb{K}^{\mathcal{P}}$ converge if and only if they are "pointwise eventually constant." That is, a sequence $\{\mathbf{f}_n\}_{n\geq 0}$ of elements of $\mathbb{K}^{\mathcal{P}}$ converges to $\mathbf{f}$ if and only if for each $x \in \mathcal{P}$, there exists $n_x \in \mathbb{N}$ such that for $n \geq n_x$, $\mathbf{f}_n(x) = \mathbf{f}(x)$.

Any $x \in \mathcal{P}$ can be associated with the unique function $\mathbf{x} \in \mathbb{K}^{\mathcal{P}}$ given by

$$\mathbf{x}(y) = \delta_{x,y} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases}$$

for any $y \in \mathcal{P}$. Now, given any sequence $\{a_x\}_{x\in\mathcal{P}}$ of elements of $\mathbb{K}$, consider the infinite sum

$$\sum_{x\in\mathcal{P}} a_x\mathbf{x}. \tag{2.1}$$

We wish to verify that this sum is an element of $\mathbb{K}^{\mathcal{P}}$, and that it is equal to the function $\mathbf{f}$ given by $\mathbf{f}(y) = a_y$. Since $\mathcal{P}$ is a countable union of the finite sets of elements of rank i, namely,

$$\mathcal{P}_i := \{x \in \mathcal{P} : \rho(x) = i\},$$

it is countable, so we can bijectively associate to each element of $\mathcal{P}$ a natural number. Let x_i denote the element of $\mathcal{P}$ corresponding to i. The sum (2.2) is the limit of the sequence of partial sums given by

$$\mathbf{f}_k := \sum_{0\leq i\leq k} a_x\mathbf{x}_i,$$

provided the limit exists. For any $y \in \mathcal{P}$, let n_y be such that $x_{n_y} = y$. Then, for any $n \geq n_y$,

$$\mathbf{f}_n(y) = \sum_{0\leq i\leq n} a_{x_i}\mathbf{x}_\mathbf{i}(y) = a_y = \mathbf{f}(y)$$

so this sequence converges to $\mathbf{f}$. We have proven the following.

Lemma 2.1. *For any* $\mathbf{f} \in \mathbb{K}^{\mathcal{P}}$,

$$\sum_{x \in \mathcal{P}} \mathbf{f}(x)\mathbf{x} = \mathbf{f}. \tag{2.2}$$

Equation (2.2) will often be written as

$$\mathbf{f} = \sum_{x \in \mathcal{P}} f_x \mathbf{x}.$$

From this, we see that $\mathbb{K}^{\mathcal{P}}$ is exactly the vector space we want to use for working with infinite linear combinations. So, for example, we may work with a sum over all poset elements, denoted by

$$\mathbf{P} := \sum_{x \in \mathcal{P}} \mathbf{x}.$$

The vector space $\mathbb{K}^{\mathcal{P}}$ has a convenient expression as a direct product of an infinite number of finite-dimensional subspaces. Define the subspace $\mathbb{K}^{\mathcal{P}_i}$ of $\mathbb{K}^{\mathcal{P}}$ by

$$\mathbb{K}^{\mathcal{P}_i} = \operatorname{span}_{\mathbb{K}}\{\mathbf{x} : x \in \mathcal{P}_i\}.$$

It is clear that $\mathcal{P}$ is the disjoint union of all the $\mathcal{P}_i$, so

$$\mathbb{K}^{\mathcal{P}} = \prod_{i \geq 0} \mathbb{K}^{\mathcal{P}_i}$$

is a direct product of finite-dimensional vector spaces.

As in Chapter 1, let

$$\mathcal{U}_x = \{y \in \mathcal{P} : x \lessdot y\},$$

$$\mathcal{D}_x = \{y \in \mathcal{P} : y \lessdot x\}$$

and, for any $x \in \mathcal{P}$, define the linear operators $U : \mathbb{K}^{\mathcal{P}} \to \mathbb{K}^{\mathcal{P}}$ and $D : \mathbb{K}^{\mathcal{P}} \to \mathbb{K}^{\mathcal{P}}$ as follows.

$$U(\mathbf{x}) = \sum_{y \in \mathcal{U}_x} \mathbf{y} = \sum_{x \lessdot y} \mathbf{y}$$

and

$$D(\mathbf{x}) = \sum_{y \in \mathcal{D}_x} \mathbf{y} = \sum_{y \lessdot x} \mathbf{y}.$$

We can extend U and D to the subspace $\mathbb{K}^{\mathcal{P}}_{\text{fin}}$ consisting of functions of finite support, *i.e.* functions $\mathbf{f}$ such that $\mathbf{f}(x) = 0$ for all but finitely many values of x, by specifying that U and D be linear.

To extend this definition to all of $\mathbb{K}^{\mathcal{P}}$, if $\mathbf{f} = \sum_{x \in \mathcal{P}} f_x \mathbf{x}$, let

$$U(\mathbf{f}) = \sum_{x \in \mathcal{P}} f_x U(\mathbf{x}) \text{ and } D(\mathbf{f}) = \sum_{x \in \mathcal{P}} f_x D(\mathbf{x}). \tag{2.3}$$

Note that if $\mathbf{f}$ has finite support, this definition reduces to the earlier one. We must verify that these operators are indeed well-defined; that is, we must verify that $U(\mathbf{f}) \in \mathbb{K}^{\mathcal{P}}$ and $D(\mathbf{f}) \in \mathbb{K}^{\mathcal{P}}$ for all $\mathbf{f} \in \mathbb{K}^{\mathcal{P}}$.

We shall use the "cover function" given by

$$c(x,y) = \begin{cases} 1 & \text{if } x \lessdot y, \\ 0 & \text{otherwise.} \end{cases}$$

This allows the operators U and D to be written in a more convenient fashion as sums over the whole poset $\mathcal{P}$, namely,

$$U(\mathbf{x}) = \sum_{y \in \mathcal{P}} c(x,y)\mathbf{y}$$

and

$$D(\mathbf{x}) = \sum_{y \in \mathcal{P}} c(y,x)\mathbf{y}.$$

Writing the operators in this form, we can see that that they are well-defined on elements of the form $\mathbf{x}$ for $x \in \mathcal{P}$ — defining $\mathbf{u}, \mathbf{d} \in \mathbb{K}^{\mathcal{P}}$ by $\mathbf{u}(y) = c(x,y)$ and $\mathbf{d}(y) = c(y,x)$ for fixed x, then $U(\mathbf{x}) = \mathbf{u}$ and $D(\mathbf{x}) = \mathbf{d}$.

Let $\mathbf{f} = \sum_{x \in \mathcal{P}} f_x \mathbf{x} \in \mathbb{K}^{\mathcal{P}}$. Observe that

$$\begin{aligned} U(\mathbf{f}) &= \sum_{x \in \mathcal{P}} f_x U(\mathbf{x}) \\ &= \sum_{x \in \mathcal{P}} \sum_{y \in \mathcal{P}} f_x c(x,y)\mathbf{y} \\ &= \sum_{y \in \mathcal{P}} \sum_{x \in \mathcal{P}} f_x c(x,y)\mathbf{y}. \end{aligned}$$

Note that the interchanging of summation order is valid because the summation over $\mathbf{y}$ is finite — $c(x,y)$ is nonzero for at most $|\mathcal{P}_{\rho(x)+1}|$ values of y, and $\mathcal{P}$ has a finite number of elements of each rank. In order to verify that $U(\mathbf{f})$ is an element of $\mathbb{K}^{\mathcal{P}}$, it suffices to check that $U(\mathbf{f})(y) \in \mathbb{K}$ for all $y \in \mathcal{P}$. Note that $U(\mathbf{f})(y) = \sum_{x \in \mathcal{P}} f_x c(x,y)$. But this is a finite sum of elements of $\mathbb{K}$, so it is in $\mathbb{K}$. Thus $U(\mathbf{f}) \in \mathbb{K}^{\mathcal{P}}$ for all $\mathbf{f} \in \mathbb{K}^{\mathcal{P}}$, so U is a well-defined endomorphism of $\mathbb{K}^{\mathcal{P}}$. D is also well-defined by a similar argument.

The following lemma justifies the use of equation (2.3) to extend the definition of U and D from vectors of finite support to all of $\mathbb{K}^{\mathcal{P}}$. The extension of the definition is natural in the sense that it forces the operators U and D to be continuous.

Lemma 2.2. *U and D are continuous, that is, if $\{\mathbf{f}_n\}_{n \geq 0}$ is a sequence in $\mathbb{K}^{\mathcal{P}}$ which converges to $\mathbf{f}$, then*

$$\lim_{n \to \infty} U(\mathbf{f}_n) = U(\mathbf{f})$$

and

$$\lim_{n \to \infty} D(\mathbf{f}_n) = D(\mathbf{f}).$$

Proof: By Lemma 2.1, we can write

$$\mathbf{f} = \sum_{x \in \mathcal{P}} f_x \mathbf{x} \text{ and } \mathbf{f}_n = \sum_{x \in \mathcal{P}} f_{n,x} \mathbf{x}.$$

Fix $y \in \mathcal{P}$. As argued above,

$$U(\mathbf{f})(y) = \sum_{x \lessdot y} f_x$$

and similarly,

$$U(\mathbf{f}_n)(y) = \sum_{x \lessdot y} f_{n,x}.$$

Since $\mathbf{f}_n \to \mathbf{f}$, for each $x \in \mathcal{D}_y$, there exists n_x such that if $n \geq n_x$, then $f_{n,x} = f_x$. Let

$$n_y = \max_{x \in \mathcal{D}_x} n_x,$$

which is well-defined since $\mathcal{D}_x$ is finite. If $n \geq n_y$, then $f_{n,x} = f_x$ for all $x \lessdot y$, so $U(\mathbf{f})(y) = U(\mathbf{f}_n)(y)$. Thus, $U(\mathbf{f}_n) \to U(\mathbf{f})$, so U is continuous. The argument that D is continuous is similar. $\square$

Now, we are in a position to examine the commutator $DU - UD$ of U and D, which will lead to the definition of a differential poset. Several facts about the cover function are useful in computations involving U and D. First, $c(x,y)^2 = c(x,y)$. Second,

$$c(x,y)c(x,z) = \begin{cases} 1 & \text{if } x \lessdot y \text{ and } x \lessdot z, \\ 0 & \text{otherwise,} \end{cases}$$

or, in other words,

$$c(x,y)c(x,z) = \begin{cases} 1 & \text{if } x \in \mathcal{D}_y \cap \mathcal{D}_z, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$c(y,x)c(z,x) = \begin{cases} 1 & \text{if } x \in \mathcal{U}_y \cap \mathcal{U}_z, \\ 0 & \text{otherwise.} \end{cases}$$

The commutator of U and D may be calculated directly using the definitions of U and D. For any $x \in \mathcal{P}$,

$$\begin{aligned} UD(x) &= U\left(\sum_{y\in\mathcal{P}} c(y,x)y\right) \\ &= \sum_{y\in\mathcal{P}} c(y,x)U(y) \\ &= \sum_{y\in\mathcal{P}} c(y,x)\sum_{z\in\mathcal{P}} c(y,z)z \\ &= \sum_{z\in\mathcal{P}}\sum_{y\in\mathcal{P}} c(y,x)c(y,z)z \\ &= \sum_{z\in\mathcal{P}} |\mathcal{D}_x \cap \mathcal{D}_z| z. \end{aligned}$$

Similarly,

$$DU(x) = \sum_{z\in\mathcal{P}} |\mathcal{U}_x \cap \mathcal{U}_z| z,$$

so

$$(DU - UD)(x) = \sum_{z\in\mathcal{P}} \left(|\mathcal{U}_x \cap \mathcal{U}_z| - |\mathcal{D}_x \cap \mathcal{D}_z|\right) z.$$

From this, it follows that in order for $DU - UD = rI$ to hold, the following conditions must be satisfied:

$$|\mathcal{U}_x \cap \mathcal{U}_y| = |\mathcal{D}_x \cap \mathcal{D}_y| \text{ for } x \neq y \tag{2.4}$$

and

$$|\mathcal{U}_x| = |\mathcal{D}_x| + r. \tag{2.5}$$

From this, we obtain the following.

Theorem 2.3 (Stanley). *Let $\mathcal{P}$ be a poset satisfying conditions 1 to 4 on page 12. (Namely, one that is locally finite, graded, has a least element $\hat{0}$ and has finitely many elements of each rank.) Then $DU - UD = rI$ if and only if equations (2.4) and (2.5) are satisfied for all $x, y \in \mathcal{P}$ with $x \neq y$.*

We are now in a position to provide two definitions of our object of study, one combinatorial and one algebraic, which are equivalent by Theorem 2.3.

Definition 2.4 (Differential Poset (Combinatorial)). *Let $\mathcal{P}$ be a locally finite, graded, partially ordered set with a least element $\hat{0}$ and finitely many elements of each rank. $\mathcal{P}$ is said to be* r**-differential** *if $|\mathcal{U}_x| = |\mathcal{D}_x| + r$ for all $x \in \mathcal{P}$, and for all $x \neq y$, $|\mathcal{U}_x \cap \mathcal{U}_y| = |\mathcal{D}_x \cap \mathcal{D}_y|$.*

Definition 2.5 (Differential Poset (Algebraic)). *Let $\mathcal{P}$ be a locally finite, graded, partially ordered set with a least element $\hat{0}$ and finitely many elements of each rank. $\mathcal{P}$ is said to be* r**-differential** *if the operators $U, D : \mathbb{K}^{\mathcal{P}} \to \mathbb{K}^{\mathcal{P}}$ given by $U(\mathbf{x}) = \sum_{x \lessdot y} \mathbf{y}$ and $D(\mathbf{x}) = \sum_{y \lessdot x} \mathbf{y}$ satisfy $DU - UD = rI$.*

Young's Lattice $\mathcal{Y}$ is an example of a 1-differential poset — the proofs of (1.2) and (1.3) in Chapter 1 demonstrate that $\mathcal{Y}$ satisfies Definition 2.4.

In the case of a lattice, condition (2.4) is often easy to verify.

Proposition 2.6 (Stanley). *Let $\mathcal{P}$ be a lattice that satisfies conditions 1 to 4 on page 12. (Namely, it is locally finite, graded, has a least element $\hat{0}$ and has finitely many elements of each rank.) If $\mathcal{P}$ satisfies condition (2.5), then $\mathcal{P}$ is r-differential if and only if $\mathcal{P}$ is modular.*

Proof: $\mathcal{P}$ is r-differential if and only if it satisfies condition (2.4). By Lemma A.2, $\mathcal{P}$ is modular if and only if the condition

$$x, y \lessdot x \vee y \Leftrightarrow x \wedge y \lessdot x, y$$

is satisfied for all $x \neq y$ in $\mathcal{P}$. Note that if $a \in \mathcal{U}_x \cap \mathcal{U}_y$, then $a = x \vee y$, and if $b \in \mathcal{D}_x \cap \mathcal{D}_y$, then $b = x \wedge y$. In particular, such a and b are unique — that is, $\mathcal{U}_x \cap \mathcal{U}_y$ (resp. $\mathcal{D}_x \cap \mathcal{D}_y$) is either empty or consists of the single element

$x \vee y$ (resp. $x \wedge y$). Thus, condition (2.4) is equivalent to the modularity condition, proving the result. $\square$

In light of this proposition, to prove that $\mathcal{Y}$ is 1-differential, the verification of (1.2) is unnecessary, since $\mathcal{Y}$ is modular.

2.1.1 Action of D on $\mathbf{P}$

For the purposes of enumerating walks, an important, and very useful consequence of the combinatorial definition of a differential poset is the following.

Theorem 2.7. *Let $\mathcal{P}$ be a r-differential poset. Then $D\mathbf{P} = (U + r)\mathbf{P}$.*

Proof: Note that

$$D\mathbf{P} = \sum_{x\in\mathcal{P}} D\mathbf{x} = \sum_{x\in\mathcal{P}}\sum_{y\in\mathcal{P}} c(y,x)\mathbf{y} = \sum_{y\in\mathcal{P}}\sum_{x\in\mathcal{P}} c(y,x)\mathbf{y}.$$

Similarly, $U\mathbf{P} = \sum_{y\in\mathcal{P}}\sum_{x\in\mathcal{P}} c(x,y)\mathbf{y}$. Combining these two equations,

$$\begin{aligned}(D-U)\mathbf{P} &= \sum_{y\in\mathcal{P}}\left(\sum_{x\in\mathcal{P}} c(y,x) - \sum_{x\in\mathcal{P}} c(x,y)\right)\mathbf{y}\\ &= \sum_{y\in\mathcal{P}} \left(|\mathcal{U}_y| - |\mathcal{D}_y|\right)\mathbf{y}\\ &= \sum_{y\in\mathcal{P}} r\mathbf{y}\\ &= r\mathbf{P}.\end{aligned}$$

Note that the penultimate equality uses the combinatorial Definition 2.4 of a differential poset. Hence $D\mathbf{P} = (U + r)\mathbf{P}$. $\square$

2.1.2 On Examples of Differential Posets

It should be noted that Young's Lattice is not the only example of a differential poset. Stanley defines an r-differential poset called the Fibonacci differential poset, denoted by $Z(r)$, in [17]. The ground set of $Z(r)$ is the set of all sequences from $\{a_1, \ldots a_r, b\}^*$. The order on $Z(r)$ can be given by specifying the cover relations, and extending by transitivity. Define $w \lessdot w'$ in $Z(r)$ if and only if either

1. for some b in the sequence w' such that no letters to the left of it are of the form a_i, changing that b to some a_i results in w, or

2. deleting the first letter of the form a_i in w' results in w.

Elementary arguments (such as those given by Stanley) can be used to show that $Z(r)$ satisfies the combinatorial definiton of an r-differential poset. The motivation for the name of this poset is that in the case $r = 1$, the sizes of level sets are Fibonacci numbers; that is, the rank-generating series is $F(Z(r); q) = (1-q-q^2)^{-1}$. Though all examples presented in the remainder of this chapter deal with walk enumeration on Young's Lattice, they can easily be re-written as examples of walk enumeration on $Z(r)$.

Though there is an infinite class of differential posets — Stanley's paper contains a construction due to Wagner — Young's Lattice and the Fibonacci differential poset are the only major examples of interest. Indeed, one open problem posed by Stanley is to find other "interesting" examples of r-differential posets. A more general context than differential posets, which admits many more examples, is discussed in Chapter 3.

2.2 The Combinatorics-to-Algebra Dictionary

The purpose of this section is to examine in more detail the connections between the algebra and combinatorics of differential posets. Our ultimate goal is to enumerate Hasse walks of a fixed shape W, starting and ending at specified poset elements, where W is a monomial in U and D. To do this, we shall derive an algebraic formula which, when evaluated, gives the number of walks of shape W between x and y.

In any poset $\mathcal{P}$, the vector space $\mathbb{K}^{\mathcal{P}}$ may be used to encode a subset $S \subset \mathcal{P}$ as the function that is 1 on S and 0 elsewhere, namely

$$\mathbf{S} = \sum_{x \in S} \mathbf{x}.$$

(The alternative notation $\mathbf{S} = \underline{S}$ may be used when the bold face causes ambiguity.) Conversely, any function $\mathbf{f} \in \mathbb{K}^{\mathcal{P}}$ which takes values in $\{0, 1\}$ encodes the subset of $\mathcal{P}$ on which $\mathbf{f}$ is equal to 1. More generally, if $M \subset \mathcal{P}$ is a multiset, where $x \in M$ with multiplicity m_x, then M may be encoded as the function given by $\mathbf{M}(x) = m_x$, that is,

$$\mathbf{M} = \sum_{x \in \mathcal{P}} m_x \mathbf{x}.$$

Conversely, it is clear that any non-negative integer-valued function in $\mathbb{K}^{\mathcal{P}}$ encodes a multiset of elements of $\mathcal{P}$. Note that if A and B are multisets, then $\mathbf{A} + \mathbf{B} = \underline{A \cup B}$ is a vector which encodes the multiset $A \cup B$.

In order to obtain the multiplicity of x as an element of M, we simply evaluate $\mathbf{M}(x)$. Let $\langle \cdot, \cdot \rangle : \mathcal{P} \times \mathbb{K}^{\mathcal{P}} \to \mathbb{K}$ be given by

$$\langle x, \mathbf{f} \rangle = \mathbf{f}(x).$$

This functional will be applied to algebraic objects, producing numbers which may be given a combinatorial interpretation. Note that this function is linear in the second variable. The function $\langle x, \mathbf{f} \rangle = \mathbf{f}(x)$ may be extended linearly to $\mathbb{K}^{\mathcal{P}}_{\text{fin}}$ in the first variable, as follows.

$$\begin{aligned} \langle \mathbf{g}, \mathbf{f} \rangle &= \left\langle \sum_{x \in \mathcal{P}} g_x \mathbf{x}, \mathbf{f} \right\rangle \\ &= \sum_{x \in \mathcal{P}} g_x \langle \mathbf{x}, \mathbf{f} \rangle \\ &= \sum_{x \in \mathcal{P}} g_x \mathbf{f}(x). \end{aligned}$$

This sum is well-defined because $g_x = 0$ for all but finitely many $x \in \mathcal{P}$. Note that we may not extend this function to $\mathbb{K}^{\mathcal{P}}$ in the first variable, since, in general, the above computation would result in a sum which does not have finite support.

Note that for $x, y \in \mathcal{P}$, this bilinear functional satisfies

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}(x) = \delta_{x,y},$$

so this is analogous to the standard inner product on a finite-dimensional vector space. In particular,

$$\begin{aligned} \langle \mathbf{g}, \mathbf{f} \rangle &= \sum_{x,y \in \mathcal{P}} g_x f_y \delta_{x,y} \\ &= \sum_{x \in \mathcal{P}} g_x f_x. \end{aligned}$$

Note that if $\mathbf{f}$ and $\mathbf{g}$ both have finite support, then $\langle \mathbf{f}, \mathbf{g} \rangle = \langle \mathbf{g}, \mathbf{f} \rangle$. Thus, its restriction to a finite-dimensional subspace such as $\mathbb{K}^{\mathcal{P}_n}$ is in fact an inner product.

Next, we consider combinatorial interpretations of the operator U. In terms of the notation introduced in the preceding paragraphs, for any $x \in \mathcal{P}$, $U(x) = \underline{\mathcal{U}_x}$. If M is any multiset of elements of $\mathcal{P}$, then the linearity of U allows us to write

$$U(\mathbf{M}) = U\left(\sum_{x \in \mathcal{P}} m_x \mathbf{x}\right) = \sum_{x \in \mathcal{P}} m_x U(\mathbf{x}) = \sum_{x \in \mathcal{P}} m_x \underline{\mathcal{U}_x} = \underline{\cup_{x \in M} \mathcal{U}_x},$$

where the union in the rightmost expression is a multiset union. So, the effect of U on a vector which encodes a multiset M is to produce a vector which encodes the multiset obtained from M by replacing each element with the set of elements immediately above it in the partial order.

Consider Young's lattice as an example. Let $M = \{(3), (21)\} \subset \mathcal{Y}$, which corresponds to the algebraic object $\mathbf{M} = (3)+(21)$. Applying U to this gives

$$U(\mathbf{M}) = U(3)+U(21) = [(31)+(4)]+[(211)+(22)+(31)] = (211)+(22)+2(31)+(4).$$

But this corresponds to the multiset $\{(211), (22), (31), (31), (4)\}$, which is exactly what we obtain if we replace each element $x \in M$ by $\mathcal{U}_x$. (The discussion for the operator D is similar.)

Recall that the shape of a Hasse walk may be encoded as a monomial $W \in \{U, D\}^*$. Given sets $A, B \subseteq \mathcal{P}$ with B finite, and $W \in \{U, D\}^*$, let $\alpha(A \xrightarrow{W} B)$ denote the number of walks of shape W which start in A and end in B. With this terminology in place, we can now state the main result of this section. (This result is used implicitly by Stanley in [17].)

Lemma 2.8. *Let A and B be subsets of $\mathcal{P}$, with B finite. Let $W \in \{U, D\}^*$. Then*

$$\alpha(A \xrightarrow{W} B) = \langle \mathbf{B}, W\mathbf{A} \rangle.$$

Proof: Use induction on the length of W. If W is the empty word, then for any $x \in B$, $y \in A$,

$$\langle \mathbf{x}, W\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

which is the number of walks from x to y of length 0. Using bilinearity of $\langle \cdot, \cdot \rangle$, we find that $\langle \mathbf{B}, W\mathbf{A} \rangle$ is the number of walks from A to B of length 0, so the lemma holds in the base case.

Suppose that the length of W is at least one, and that the lemma holds for walks of shorter length. Note that either $W = W'U$ or $W = W'D$ for some monomial W' of length shorter than W. Let $x \in B$ and $y \in A$. Then

$$\begin{aligned} \langle \mathbf{x}, W\mathbf{y} \rangle &= \langle \mathbf{x}, W'U\mathbf{y} \rangle \\ &= \langle \mathbf{x}, W'\underline{\mathcal{U}_y} \rangle. \end{aligned}$$

By induction, this is the number of walks from $\mathcal{U}_y$ to x of shape W'. Note that there is a bijection between these walks and walks from y to x of shape $W = W'U$. So $\langle \mathbf{x}, W\mathbf{y} \rangle$ is the number of walks from y to x of shape W. Extending linearly, $\langle \mathbf{B}, W\mathbf{A} \rangle$ is the number of walks from A to B of shape W. □

If the sets A and B happen to be level subsets of the differential poset, then the notation

$$\alpha(n \xrightarrow{W} n + \ell(W)) = \langle \mathbf{P}_{n+\ell(W)}, W\mathbf{P}_n \rangle$$

is used. That is, $\alpha(n \xrightarrow{W} n + \ell(W))$ is the number of Hasse walks of shape W starting at an element of rank n and ending at rank $n + \ell(W)$. Furthermore, when $W = U^{m-n}$, we shall often write

$$\alpha(n \to m) = \langle \mathbf{P}_m, U^{m-n}\mathbf{P}_n \rangle.$$

One final observation about the operators U and D is the following. For $x, y \in \mathcal{P}$,

$$\begin{aligned} \langle \mathbf{x}, U\mathbf{y} \rangle &= \langle \mathbf{x}, \underline{\mathcal{U}_y} \rangle \\ &= \begin{cases} 1 & \text{if } y \lessdot x, \\ 0 & \text{otherwise} \end{cases} \\ &= \langle \underline{\mathcal{D}_x}, \mathbf{y} \rangle \\ &= \langle D\mathbf{x}, \mathbf{y} \rangle. \end{aligned}$$

Similarly, $\langle \mathbf{x}, D\mathbf{y} \rangle = \langle U\mathbf{x}, \mathbf{y} \rangle$. So U and D are adjoint with respect to this bilinear functional. Let A^* denote the adjoint of the operator A. Then, for any $x, y \in \mathcal{P}$,

$$\begin{aligned} \langle \mathbf{x}, W\mathbf{y} \rangle &= \langle W^*\mathbf{x}, \mathbf{y} \rangle \\ &= \langle \mathbf{y}, W^*\mathbf{x} \rangle. \end{aligned}$$

Combinatorial object	**Algebraic object**
The poset $\mathcal{P}$	The vector space $\mathbb{K}^{\mathcal{P}}$
Multiset $M \subset \mathcal{P}$	The vector $\mathbf{M} \in \mathbb{K}^{\mathcal{P}}$
"Up" step in a walk	The operator U
"Down" step in a walk	The operator D
$\alpha(A \xrightarrow{W} B)$	$\langle \mathbf{B}, W\mathbf{A} \rangle$
A traversal of a walk in reverse direction	The adjoint of an operator

Table 2.1: Connection between algebra and combinatorics in a differential poset

Hence, the number of walks from y to x of shape W is equal to the number of walks from x to y of shape W^*. Note that W^* is obtained from W by writing W in reverse order, and interchanging D and U. From this we see that the operation of taking adjoints of operators corresponds to traversing walks backwards, from which the relation $\alpha(y \stackrel{W}{\to} x) = \alpha(x \stackrel{W^*}{\to} y)$ becomes obvious from a combinatorial point of view.

The results of this section are summarized in Table 2.1. Having reduced the problem of enumerating walks from A to B of shape W to the problem of evaluating $\langle \mathbf{A}, W\mathbf{B} \rangle$, we now turn our attention to the algebraic problem of evaluating this expression.

2.3 Algebraic Results for Differential Posets

In order to obtain any interesting results about differential posets, we must be able to manipulate expressions involving U and D. This section introduces a number of results that may be used to simplify endomorphisms built up from the operators U and D. The results of this section were first proven by Stanley [17].

2.3.1 Endomorphisms of $\mathbb{K}^{\mathcal{P}}$

We first verify that the expressions we work with are in fact endomorphisms of the vector space $\mathbb{K}^{\mathcal{P}}$. Since U and D are endomorphisms of $\mathbb{K}^{\mathcal{P}}$, it is clear that any polynomial in the noncommuting U and D corresponds to an endomorphism of $\mathbb{K}^{\mathcal{P}}$. However, the question becomes less clear for formal power series involving U and D. Examples of such series that do not correspond to endomorphisms of $\mathbb{K}^{\mathcal{P}}$ may be found by considering certain infinite sets of walks. Consider the set of all decreasing walks ending at $\hat{0}$. There is an infinite number of such walks, so any attempt to encode this as a vector will not result in an element of $\mathbb{K}^{\mathcal{P}}$, since evaluating the vector at $\hat{0}$ would give infinity. Thus, any transformation resulting in a vector encoding the set of all decreasing walks ending at $\hat{0}$ cannot be an endomorphism. This transformation must be $f(D) = \sum_{i\geq 0} D^i = \frac{1}{1-D}$, and applying this to the vector $\mathbf{P}$ should create a "vector" enumerating all decreasing walks, so $\frac{1}{1-D}$ is not an endomorphism of $\mathbb{K}^{\mathcal{P}}$.

This discussion may be made more rigorous as follows. Note that for any $x \in \mathcal{P}$,

$$\langle \hat{0}, (1-D)^{-1}x\rangle = \sum_{n\geq 0}\langle \hat{0}, D^n x\rangle = \langle \hat{0}, D^{\rho(x)}x\rangle \geq 1.$$

Extending this linearly to the element $\mathbf{P} = \sum_{x\in P} x$, we find that $\langle \hat{0}, f(D)\mathbf{P}\rangle = \infty$, *i.e.* the coefficient of $\hat{0}$ in $f(D)\mathbf{P}$ is infinite, so $f(D)\mathbf{P}$ is not an element of $\mathbb{K}^{\mathcal{P}}$. Thus, the element $1 - D$ does not have an inverse in the ring of endomorphisms of $\mathbb{K}^{\mathcal{P}}$.

There are two important classes of formal power series in U and D which are indeed endomorphisms of $\mathbb{K}^{\mathcal{P}}$. These are dealt with in the following lemmas.

Lemma 2.9 (Stanley). *Let $f(z) \in \mathbb{K}[[z]]$. Then $f(U)$ is a well-defined endomorphism of $\mathbb{K}^{\mathcal{P}}$.*

Proof: Given any $\mathbf{y} \in \mathbb{K}^{\mathcal{P}}$, it suffices to show that

$$[f(U)\mathbf{y}](x) = \langle \mathbf{x}, f(U)\mathbf{y}\rangle < \infty$$

for all $x \in \mathcal{P}$. Write $f(U) = \sum_{n\geq 0} a_n U^n$ and observe that for any n,

$$U^n\mathbf{y} \in \prod_{i\geq n} \mathbb{K}^{\mathcal{P}_i}$$

since U maps from $\mathbb{K}^{\mathcal{P}_i}$ to $\mathbb{K}^{\mathcal{P}_{i+1}}$. Thus, if $n > \rho(x)$, $\langle x, U^n y\rangle = 0$. Hence

$$\langle \mathbf{x}, f(U)\mathbf{y}\rangle = \sum_{n\geq 0} a_n\langle \mathbf{x}, U^n\mathbf{y}\rangle = \sum_{0\leq n\leq\rho(x)} a_n\langle \mathbf{x}, U^n\mathbf{y}\rangle.$$

Since this sum is finite for all $x \in P$, $f(U)\mathbf{y}$ is well-defined for all $\mathbf{y} \in \mathbb{K}^{\mathcal{P}}$. Thus $f(U)$ is an endomorphism of $\mathbb{K}^{\mathcal{P}}$. $\square$

Lemma 2.10 (Stanley). *Let $g(z,q) \in \mathbb{K}[[z,q]]$, where $\mathbb{K}$ is a field containing $\mathbb{Q}((t))$. Then $g(U,Dt)$ is a well-defined endomorphism of $\mathbb{K}^{\mathcal{P}}$.*

Proof: Let $\mathbf{y} \in \mathbb{K}^{\mathcal{P}}$ and $x \in \mathcal{P}$. Write $g(z,q) = \sum_{n\geq 0} a_n(z)q^n$ for $a_n \in \mathbb{K}[[z]]$. Then

$$\begin{aligned}\langle \mathbf{x}, g(U,Dt)\mathbf{y}\rangle &= \left\langle \mathbf{x}, \sum_{n\geq 0} a_n(U)D^n t^n \mathbf{y}\right\rangle \\ &= \sum_{n\geq 0} t^n \langle \mathbf{x}, A_n(U)D^n\mathbf{y}\rangle .\end{aligned}$$

By Lemma 2.9, $\langle \mathbf{x}, A_n(U)D^n\mathbf{y}\rangle \in \mathbb{K}$, so $\langle \mathbf{x}, g(U,Dt)\mathbf{y}\rangle \in \mathbb{K}[[t]] = \mathbb{K}$. $\square$

2.3.2 Differential Action of D

In any r-differential poset, the equation $DU = UD + rI$ holds. This equation may be viewed as a way dealing with expressions in U and D by "moving D to the right," as demonstrated in the following.

Example 2.11.

$$\begin{aligned}DU^2 &= (UD + rI)U \\ &= UDU + rU \\ &= U(UD + rI) + rU \\ &= 2rU + U^2D.\end{aligned}$$

It is clear that this technique may be applied repeatedly. The result is a generalization of the equation $DU = UD + rI$ to deal with the situation in which D is applied to any formal power series in U. (We know that any such series gives an endomorphism of $\mathbb{K}^{\mathcal{P}}$ by Lemma 2.9.)

Theorem 2.12 (Stanley). *Let $g(z) \in \mathbb{K}[[z]]$. Then in any r-differential poset,*

$$Dg(U) = r\frac{\partial}{\partial U}g(U) + g(U)D.$$

Proof: First, we prove by induction on n that $DU^n = rnU^{n-1} + U^nD$. The base case, when $n = 1$, is $DU = r + UD$, which holds by Definition 2.5

in a r-differential poset. Suppose that $DU^{n-1} = r(n-1)U^{n-2} + U^{n-1}D$. Then

$$\begin{aligned} DU^n &= (DU^{n-1})U \\ &= (r(n-1)U^{n-2} + U^{n-1}D)U \\ &= r(n-1)U^{n-1} + U^{n-1}(UD + rI) \\ &= rnU^{n-1} + U^nD. \end{aligned}$$

Use linearity to extend this result to $g(U)$ for any $g(z) \in \mathbb{K}[[z]]$, to achieve the desired result. $\square$

This theorem justifies the use of the term "differential poset," since it implies that D behaves like the differential operator $\frac{\partial}{\partial U}$. When we apply the result of this theorem to $\mathbf{P}$, the sum over all poset elements, we can use Theorem 2.7 to obtain

$$\begin{aligned} Dg(U)\mathbf{P} &= \left(r\frac{\partial}{\partial U}g(U) + g(U)D\right)\mathbf{P} \\ &= \left(r\frac{\partial}{\partial U}g(U) + g(U)(U+r)\right)\mathbf{P}. \end{aligned}$$

Thus, we obtain the following corollary.

Corollary 2.13 (Stanley). *In an r-differential poset, for any $g(z) \in \mathbb{K}[[z]]$,*

$$Dg(U)\mathbf{P} = \left(r\frac{\partial}{\partial U}g(U) + (U+r)g(U)\right)\mathbf{P}.$$

Another corollary of Theorem 2.12 is the following, which allows us to compute repeated application of D.

Corollary 2.14. *In an r-differential poset, for any $g(z) \in \mathbb{K}[[z]]$,*

$$D^n g(U) = \sum_{0 \le i \le n} \binom{n}{i} r^i \frac{\partial^i g}{\partial U^i} D^{n-i}.$$

Proof: Use induction on n. For the base case $n = 1$, use Theorem 2.12.

Suppose that $n \geq 2$ and that the result holds for $n-1$. Then

$$\begin{aligned}
D^n g(U) &= D(D^{n-1}g(U)) \\
&= D \sum_{0\leq i\leq n-1} \binom{n-1}{i} r^i \frac{\partial^i g}{\partial U^i} D^{n-1-i} \\
&= \sum_{0\leq i\leq n-1} \binom{n-1}{i} r^i \left(r\frac{\partial^{i+1} g}{\partial U^{i+1}} + \frac{\partial^i g}{\partial U^i} D \right) D^{n-1-i} \\
&= \sum_{1\leq i\leq n} \binom{n-1}{i-1} r^i \frac{\partial^i g}{\partial U^i} D^{n-i} + \sum_{0\leq i\leq n} \binom{n-1}{i} r^i \frac{\partial^i g}{\partial U^i} D^{n-i} \\
&= \sum_{0\leq i\leq n} \binom{n}{i} r^i \frac{\partial^i g}{\partial U^i} D^{n-i}.
\end{aligned}$$

□

2.3.3 Expressing (U, D)-words as polynomials in U

Let $W \in \{U, D\}^*$ be any monomial in U and D. By applying Corollary 2.13 to the rightmost occurrence of D, and repeating for each D, we will eventually end up with a polynomial in U. This is best illustrated by an example.

Example 2.15.

$$\begin{aligned}
UUDDU\mathbf{P} &= UUD(r + U(U+r))\mathbf{P} \\
&= UUD(r + rU + U^2)\mathbf{P} \\
&= UU(r(r+2U) + (r + rU + U^2)(U+r))\mathbf{P} \\
&= (U^5 + 2rU^4 + (r^2+3r)U^3 + 2r^2U^2)\mathbf{P}.
\end{aligned}$$

The notion that we can do this to any (U, D)-word is captured in the following theorem.

Theorem 2.16 (Stanley). *Let $\mathcal{P}$ be an r-differential poset, and let $W \in \{U, D\}^*$. Then there exists a unique polynomial $p_W(z) \in \mathbb{K}[z]$ such that*

$$W\mathbf{P} = p_W(U)\mathbf{P}.$$

In fact, the polynomial p_W is defined recursively by

$$p_I(z) = 1, \tag{2.6}$$
$$p_{UV}(z) = zp_V(z), \tag{2.7}$$
$$p_{DV}(z) = rp'_V(z) + (z + r)p_V(z), \tag{2.8}$$

where $V \in \{U, D\}^*$.

Proof: Let $W \in \{U, D\}^*$ and proceed by induction on the length $|W|$ of W. If $|W| = 0$, then $W = I$, and it is clear that $\mathbf{P} = p_I(U)\mathbf{P}$. Suppose that $|W| \geq 1$ and that the theorem holds for all words of shorter length. Note that either $W = UV$ or $W = DV$, where V is a word of length $|W| - 1$. By the inductive hypothesis, there exists p_V such that

$$V\mathbf{P} = p_V(U)\mathbf{P}.$$

If $W = UV$, apply U on the left to obtain

$$UV\mathbf{P} = Up_V(U)\mathbf{P} = p_{UV}(U)\mathbf{P}.$$

Thus, $p_{UV} = zp_V$ is a polynomial satisfying the desired condition. If $W = DV$, apply D on the left to obtain

$$\begin{aligned} DV\mathbf{P} &= Dp_V(U)\mathbf{P} \\ &= (rp'_V(U) + (r + U)p_V(U))\mathbf{P} \\ &= p_{DV}(U)\mathbf{P}, \end{aligned}$$

by applying Corollary 2.13. Thus, $p_{DV} = rp'_V + (r + z)p_V$ is a polynomial satisfying the desired condition. Hence, by induction, such a polynomial exists with the formula given by equations (2.6) to (2.8).

For the uniqueness part of the theorem, suppose there are two polynomials p_W and q_W such that $W\mathbf{P} = p_W(U)\mathbf{P} = q_W(U)\mathbf{P}$. Then $(p_W - q_W)(U)\mathbf{P} = 0$. Thus, it suffices to show that for any $f(z) \in \mathbb{K}[[z]]$, if $f(U)\mathbf{P} = 0$ then $f = 0$. Suppose $f \neq 0$. Then $f(z) = z^n g(z)$ for some $n \geq 0$ and g satisfying $g(0) \neq 0$. Note that

$$\begin{aligned} \langle \mathbf{P}_n, f(U)\mathbf{P} \rangle &= \langle \mathbf{P}_n, g(U)U^n(\hat{0} + \mathbf{P} - \hat{0}) \rangle \\ &= \langle \mathbf{P}_n, g(U)U^n\hat{0} \rangle + \langle \mathbf{P}_n, g(U)U^n(\mathbf{P} - \hat{0}) \rangle \\ &= \langle \mathbf{P}_n, g(U)U^n\hat{0} \rangle, \end{aligned}$$

since $\mathbf{P} - \hat{0} \in \prod_{i \geq 1} \mathbb{K}^{\mathcal{P}_i}$, hence $g(U)U^n(\mathbf{P} - \hat{0}) \in \prod_{i \geq n+1} \mathbb{K}^{\mathcal{P}_i}$. Furthermore, $U^n\hat{0}$ is a nonzero element of $\mathbb{K}^{\mathcal{P}_n}$, so

$$\begin{aligned}
\langle \mathbf{P}_n, g(U)U^n\hat{0} \rangle &= \langle \mathbf{P}_n, g(0)U^n\hat{0} \rangle \\
&= g(0)\langle \mathbf{P}_n, U^n\hat{0} \rangle \\
&\neq 0,
\end{aligned}$$

hence $\langle \mathbf{P}_n, f(U)\mathbf{P} \rangle \neq 0$, so $f(U)\mathbf{P} \neq 0$. □

Since any formal power series in U and D is a linear combination of monomials, this result can be extended using linearity to any such series, provided it is a well-defined endomorphism of $\mathbb{K}^{\mathcal{P}}$.

Corollary 2.17 (Stanley). *Let $f(U, D)$ be a formal power series in U and D which defines an endomorphism of $\mathbb{K}^{\mathcal{P}}$. Then there exists a unique formal power series $p_f(z)$ such that:*

$$f(U, D)\mathbf{P} = p_f(U)\mathbf{P}.$$

Example 2.18. As an example of an application of Theorem 2.16, consider the construction of the polynomial p_{UUDDU}.

$$\begin{aligned}
p_\emptyset &= 1, \\
p_U &= z, \\
p_{DU} &= rp'_U + (r + z)p_U \\
&= r + rz + z^2, \\
p_{DDU} &= rp'_{DU} + (r + z)p_{DU} \\
&= r(r + 2z) + (r + z)(r + rz + z^2) \\
&= z^3 + 2rz^2 + (r^2 + 3r)z + 2r^2, \\
p_{UUDDU} &= z^2 p_{DDU} \\
&= z^5 + 2rz^4 + (r^2 + 3r)z^3 + 2r^2z^2.
\end{aligned}$$

As expected, this agrees with the result obtained in Example 2.15.

Equation (2.8) may also be written in the following form, by multiplying by the integrating factor $\exp(z + \frac{z^2}{2r})$.

$$p_{DV}(z) = \exp\left(-z - \frac{z^2}{2r}\right) r \frac{d}{dz}\left(\exp\left(z + \frac{z^2}{2r}\right) p_V(z)\right). \tag{2.9}$$

Equation (2.7) may be written in a similar form.

$$p_{UV}(z) = \exp\left(-z - \frac{z^2}{2r}\right) z \left(\exp\left(z + \frac{z^2}{2r}\right) p_V(z)\right). \tag{2.10}$$

Applying these equations repeatedly to a (U, D)-word, we obtain the following.

Corollary 2.19 (Stanley). *Let $W = W(U, D)$ be a monomial in U and D. Then*

$$p_W(z) = \exp\left(-z - \frac{z^2}{2r}\right) W\left(z, r\frac{d}{dz}\right) \exp\left(z + \frac{z^2}{2r}\right). \tag{2.11}$$

Proof: As before, use induction on the length of W. If $W = I$, then this corollary is clear. If $W = DV$ for some V of length $\ell(W) - 1$, then applying equation (2.9) and induction we have

$$\begin{aligned}
p_W(z) &= \exp\left(-z - \frac{z^2}{2r}\right) r\frac{d}{dz}\left(\exp\left(z + \frac{z^2}{2r}\right) p_V(z)\right) \\
&= \exp\left(-z - \frac{z^2}{2r}\right) r \\
&\quad \times \frac{d}{dz}\left(\exp\left(z + \frac{z^2}{2r}\right) \exp\left(-z - \frac{z^2}{2r}\right) V\left(z, r\frac{d}{dz}\right) \exp\left(z + \frac{z^2}{2r}\right)\right) \\
&= \exp\left(-z - \frac{z^2}{2r}\right) r\frac{d}{dz} V\left(z, r\frac{d}{dz}\right) \exp\left(z + \frac{z^2}{2r}\right) \\
&= \exp\left(-z - \frac{z^2}{2r}\right) W\left(z, r\frac{d}{dz}\right) \exp\left(z + \frac{z^2}{2r}\right).
\end{aligned}$$

A similar result is obtained in the case $W = UV$. □

Example 2.20. As a check, this corollary may be applied to $W = UUDDU$.

$$\begin{aligned}
p_{UUDDU}(z) &= \exp\left(-z - \frac{z^2}{2r}\right) z^2 r^2 \frac{d^2}{dz^2} z \exp\left(z + \frac{z^2}{2r}\right) \\
&= \exp\left(-z - \frac{z^2}{2r}\right) z^2 r \frac{d}{dz}(r + z(r+z)) \exp\left(z + \frac{z^2}{2r}\right) \\
&= \exp\left(-z - \frac{z^2}{2r}\right) z^2 (r^2 + 2rz + (r + zr + z^2)(r+z)) \exp\left(z + \frac{z^2}{2r}\right) \\
&= z^5 + 2rz^4 + (r^2 + 3r)z^3 + 2r^2 z^2,
\end{aligned}$$

agreeing with the result obtained in Examples 2.15 and 2.18.

Example 2.21. Now, suppose we want to compute the number of walks of shape $UUDDU$ from the n^{th} rank to the $n+1^{\text{th}}$ rank. By Proposition 2.8, $\alpha(n \xrightarrow{W} n+1) = \langle \mathbf{P}_{n+1}, UUDDU\mathbf{P}_n \rangle = \langle \mathbf{P}_{n+1}, UUDDU\mathbf{P} \rangle$, so

$$\begin{aligned}
\alpha(n \xrightarrow{W} n+1) &= \langle \mathbf{P}_{n+1}, (U^5 + 2rU^4 + (r^2+3r)U^3 + 2r^2U^2)\mathbf{P} \rangle \\
&= \langle \mathbf{P}_{n+1}, U^5\mathbf{P} \rangle + 2r\langle \mathbf{P}_{n+1}, U^4\mathbf{P} \rangle \\
&\quad + (r^2+3r)\langle \mathbf{P}_{n+1}, U^3\mathbf{P} \rangle + 2r^2\langle \mathbf{P}_{n+1}, U^2\mathbf{P} \rangle \\
&= \alpha(n-4 \to n+1) + 2r\alpha(n-3 \to n+1) \\
&\quad + (r^2+3r)\alpha(n-2 \to n+1) + 2r^2\alpha(n-1 \to n+1).
\end{aligned}$$

The computation of Example 2.21 is a special case of the following, where $p_W = \sum_{i\geq 0} a_i z^i$.

$$\begin{aligned}
\alpha(n \xrightarrow{W} n+k) &= \langle \mathbf{P}_{n+k}, W\mathbf{P} \rangle \\
&= \sum_{i\geq 0} a_i \langle \mathbf{P}_{n+k}, U^i\mathbf{P} \rangle \\
&= \sum_{i\geq 0} a_i \alpha(n+k-i \to n+k).
\end{aligned}$$

This result is summarized in the following.

Lemma 2.22. *Let $\mathcal{P}$ be an r-differential poset, and $W \in \{U,D\}^*$ be a word of length k. If $p_W(z) = \sum_{i\geq 0} a_i z^i$, then*

$$\alpha(n \xrightarrow{W} n+k) = \sum_{i\geq 0} a_i \alpha(n+k-i \to n+k).$$

It should now be clear that our next task is to compute the values $\alpha(n \to n+k)$ for arbitrary values of n and k.

2.4 Generating Series for $\alpha(n \to n+k)$

This section deals with the problem of enumerating monotonically increasing paths in an r-differential poset. We wish to calculate the quantity

$$\alpha(n \to n+k) = \langle \mathbf{P}_{n+k}, U^k\mathbf{P}_n \rangle = \langle D^k\mathbf{P}_{n+k}, \mathbf{P}_n \rangle = \langle \mathbf{P}_n, D^k\mathbf{P} \rangle.$$

Hence, a study of the polynomials $p_{D^k}(z)$, hereafter denoted by $p_k(z)$, is in order. By Theorem 2.16, these polynomials are defined recursively by

$$\begin{aligned} p_0(z) &= 1, \\ p_k(z) &= r\frac{\partial}{\partial z}p_{k-1}(z) + (z+r)p_{k-1}(z) \text{ for } k \geq 1. \end{aligned}$$

Using this recursion, we can derive a functional equation for the generating series $P(z,t) := \sum_{k\geq 0} p_k(z)\frac{t^k}{k!}$. Multiplying the formula by $\frac{t^k}{k!}$ and summing over all $k \geq 1$,

$$\sum_{k\geq 1} p_k(z)\frac{t^k}{k!} = \sum_{k\geq 1} r\frac{\partial}{\partial z}p_{k-1}(z)\frac{t^k}{k!} + \sum_{k\geq 1}(z+r)p_{k-1}(z)\frac{t^k}{k!}.$$

Simplifying, using the definition of $P(z,t)$ on the left and linearity of differentiation on the right,

$$P(z,t) - p_0(z) = r\frac{\partial}{\partial z}\sum_{k\geq 1} p_{k-1}(z)\frac{t^k}{k!} + (z+r)\sum_{k\geq 1} p_{k-1}(z)\frac{t^k}{k!}.$$

Differentiating this equation with respect to t, using the fact that the formal differential operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial z}$ commute, and recalling that $p_0(z) = 1$,

$$\frac{\partial P}{\partial t} = r\frac{\partial}{\partial z}\sum_{k\geq 1} p_{k-1}(z)\frac{t^{k-1}}{(k-1)!} + (z+r)\sum_{k\geq 1} p_{k-1}(z)\frac{t^{k-1}}{(k-1)!}.$$

Invoking the definition of $P(z,t)$, this simplifies to the partial differential equation

$$\frac{\partial P}{\partial t} = r\frac{\partial P}{\partial z} + (z+r)P. \tag{2.12}$$

with initial condition given by $P(z,0) = p_0(z) = 1$. This partial differential equation may be solved (say, using the method of characteristics discussed in Appendix B); however, it is possible to derive a different recursive description of the polynomials p_k, which leads to a simpler ordinary differential equation. Apply Corollary 2.14 to obtain

$$\begin{aligned} p_k(U)\mathbf{P} &= D^k\mathbf{P} \\ &= D^{k-1}(U+r)\mathbf{P} \\ &= [(U+r)D^{k-1} + (k-1)rD^{k-2}]\mathbf{P} \\ &= [(U+r)p_{k-1}(U) + (k-1)rp_{k-2}(U)]\mathbf{P}. \end{aligned}$$

By the uniqueness result of Theorem 2.16,

$$p_k(z) = (z+r)p_{k-1}(z) + (k-1)rp_{k-2}(z). \tag{2.13}$$

This recursive formula has initial conditions $p_0(z) = 1$ and $p_1(z) = z + r$. As before, this recursion may be used to derive a functional equation for $P(z,t)$. Multiplying the kth equation by $\frac{t^k}{k!}$ and summing over all $k \geq 2$,

$$\sum_{k\geq 2} p_k(z)\frac{t^k}{k!} = (r+z)\sum_{k\geq 2} p_{k-1}(z)\frac{t^k}{k!} + r\sum_{k\geq 2}(k-1)p_{k-2}\frac{t^k}{k!}.$$

Differentiating this equation with respect to t,

$$\begin{aligned}\frac{d}{dt}(P(z,t) - p_0 - p_1 t) &= (r+z)\sum_{k\geq 2} p_{k-1}\frac{t^{k-1}}{(k-1)!} + r\sum_{k\geq 2} p_{k-2}\frac{t^{k-1}}{(k-2)!} \\ &= (r+z)(P(z,t) - p_0) + rtP(z,t).\end{aligned}$$

Rearranging this, and using the initial conditions $p_0 = 1$ and $p_1 = z + r$, we obtain the ordinary differential equation

$$\frac{dP}{dt} = (r+z+rt)P,$$

with initial condition $P(z,0) = p_0 = 1$. This may be easily solved by dividing by P and integrating to obtain

$$\int_0^t \frac{1}{P(z,s)}\frac{dP}{ds}ds = \int_0^t (r+z+rs)ds,$$

hence,

$$\log P = (r+z)t + \frac{t^2 r}{2} + \log(P(z,0)).$$

After applying the initial condition $P(z,0) = 1$, we have proven the following result of Stanley. (The technique of using the recursion (2.13) to prove this is new.)

Theorem 2.23 (Stanley). *Let $\mathcal{P}$ be an r-differential poset over a field $\mathbb{K}$ containing t. Then*

$$P(z,t) = \exp\left(rt + zt + \frac{t^2 r}{2}\right). \tag{2.14}$$

p_0	1
p_1	$z+r$
p_2	$z^2+2rz+(r^2+r)$
p_3	$z^3+3rz^2+3(r^2+r)z+(r^3+3r^2)$
p_4	$z^4+4rz^3+6(r^2+r)z^2+4(r^3+3r^2)z+(r^4+6r^3+3r^2)$

Table 2.2: Values of the polynomials p_k

(Note, as a check, that this result also satisfies the partial differential equation (2.12).)

By extracting the coefficient of $\frac{t^k}{k!}$ from $P(z,t)$, we can compute the polynomials p_k. The first five polynomials in this sequence, computed using Maple, are shown in Table 2.2.

For any $n \geq 1$, let $\mathrm{inv}(n)$ denote the set of involutions in the symmetric group $\mathfrak{S}_n$. For $\sigma \in \mathrm{inv}(n)$, let $c(\sigma)$ denote the number of cycles of σ. The following observation allows us to find a formula for $p_k(z)$ in terms of the symmetric group.

Lemma 2.24. *Let a and b be indeterminates. Then*

$$\left[\frac{t^n}{n!}\right] \exp\left(at + b\frac{t^2}{2}\right) = \sum_{\sigma \in \mathrm{inv}(n)} a^{2c(\sigma)-n} b^{n-c(\sigma)}.$$

Proof: Any $\sigma \in \mathrm{inv}(n)$ may be regarded as an unordered set of cycles of length at most two. Marking 1-cycles with the indeterminate a and 2-cycles with b, it is clear that $\exp(at + bt^2/2)$ is the exponential generating series for $\mathrm{inv}(n)$. Thus, if $c_i(\sigma)$ denotes the number of cycles of length i, then

$$\left[\frac{t^n}{n!}\right] \exp\left(at + b\frac{t^2}{2}\right) = \sum_{\sigma \in \mathrm{inv}(n)} a^{c_1(\sigma)} b^{c_2(\sigma)}.$$

Note that

$$c_1(\sigma) + c_2(\sigma) = c(\sigma)$$

and

$$c_1(\sigma) + 2c_2(\sigma) = n,$$

so $c_1(\sigma) = 2c(\sigma) - n$ and $c_2(\sigma) = n - c$, proving the lemma. $\square$

Applying Lemma 2.24 with $a = (r+z)$ and $b = r$, we can obtain the following formula.

$$\begin{aligned} p_k(z) &= \left[\frac{t^n}{n!}\right] \exp\left((r+z)t + \frac{rt^2}{2}\right) \\ &= \sum_{\sigma \in \mathrm{inv}(n)} (r+z)^{2c(\sigma)-n} r^{n-c(\sigma)}. \end{aligned}$$

Example 2.25. To compute $p_3(z)$, note that the involutions of $\mathfrak{S}_n$ are, using cycle notation, (1)(2)(3), (12)(3), (13)(2), (23)(1). So there is one involution with three cycles, and three with two cycles. Thus,

$$\begin{aligned} p_3(z) &= (z+r)^3 + 3(z+r)r \\ &= z^3 + 3rz^2 + 3(r^2+r)z + (r^3+3r^2), \end{aligned}$$

which agrees with the result of Table 2.2.

2.4.1 Calculating $\alpha(n \to n+k)$

Equipped with the results of the previous section, we are now in a position to tackle the problem of computing $\alpha(n \to n+k)$. By Lemma 2.22,

$$\alpha(n \to n+k) = \sum_{i \geq 0} a_i \alpha(n-i \to n),$$

where $p_k(z) = \sum_{i\geq 0} a_i z^i$. Using the computation of p_k from the previous section, we obtain a sequence of recursive formulae for $\alpha(n \to n+k)$.

$$\begin{aligned} \alpha(n \to n+1) &= r\alpha(n \to n) + \alpha(n-1 \to n), \\ \alpha(n \to n+2) &= (r^2+r)\alpha(n \to n) + 2r\alpha(n-1 \to n) + \alpha(n-2 \to n), \\ \alpha(n \to n+3) &= (r^3+3r^2)\alpha(n \to n) + 3(r^2+r)\alpha(n-1 \to n) \\ &\quad + 3r\alpha(n-2 \to n) + \alpha(n-3 \to n), \\ &\vdots \end{aligned}$$

The base case for the recursion is given by $\alpha(n \to n) = |\mathcal{P}_n|$. These recursive formulae suggest looking for a functional equation for the generating series

$$G(q,t) := \sum_{n\geq 0}\sum_{k \geq 0} \alpha(n \to n+k) q^n \frac{t^k}{k!}$$

for $\alpha(n \to n+k)$.

Stanley's method for finding a functional equation for G is as follows. Using the results of the Sections 2.2 and 2.3, we may write this as

$$\begin{aligned} G(q,t) &= \sum_{n\geq 0}\sum_{k\geq 0}\langle \mathbf{P}_n, D^k\mathbf{P}\rangle q^n \frac{t^k}{k!} \\ &= \sum_{n\geq 0}\sum_{k\geq 0}\langle \mathbf{P}_n, p_k(U)\mathbf{P}\rangle q^n \frac{t^k}{k!} \\ &= \sum_{n\geq 0}\left\langle \mathbf{P}_n, \sum_{k\geq 0}\frac{t^k}{k!}p_k(U)\mathbf{P}\right\rangle q^n \\ &= \sum_{n\geq 0}\langle \mathbf{P}_n, P(U,t)\mathbf{P}\rangle q^n \\ &= \sum_{n\geq 0}\langle \mathbf{P}_n, \exp\left(rt + Ut + \frac{rt^2}{2}\right)\mathbf{P}\rangle q^n \\ &= \exp\left(rt + \frac{rt^2}{2}\right)\sum_{n\geq 0}\langle \mathbf{P}_n, \exp(Ut)\mathbf{P}\rangle q^n. \end{aligned}$$

Through a change of index, we may re-write this in terms of $G(q,t)$.

$$\begin{aligned} \sum_{n\geq 0}\langle \mathbf{P_n}, \exp(Ut)\mathbf{P}\rangle q^n &= \sum_{n\geq 0}\sum_{k\geq 0}\langle \mathbf{P_n}, U^k\mathbf{P}\rangle \frac{t^k}{k!}q^n \\ &= \sum_{k\geq 0}\sum_{n\geq 0}\alpha(n-k \to n)\frac{t^k}{k!}q^n \\ &= \sum_{k\geq 0}\sum_{m\geq -k}\alpha(m \to m+k)\frac{t^k}{k!}q^{m+k} (\text{substituting } m=n-k) \\ &= \sum_{k\geq 0}\sum_{m\geq 0}\alpha(m \to m+k)\frac{(tq)^k}{k!}q^m \\ &= G(q,qt). \end{aligned}$$

Note the penultimate equality uses the fact that $\alpha(m \to m+k) = 0$ if $m < 0$. Furthermore, from the definition of $G(q,t)$, it is clear that

$$G(q,0) = \sum_{n\geq 0}\alpha(n \to n)q^n = \sum_{n\geq 0}|\mathcal{P}_n|q^n = F(\mathcal{P};q).$$

Thus, we have proven the following.

Lemma 2.26 (Stanley). *The generating series $G(q,t)$ satisfies the functional equation*

$$G(q,t) = \exp\left(rt + \frac{rt^2}{2}\right) G(q,qt), \tag{2.15}$$

with initial condition

$$G(q,0) = F(\mathcal{P};q), \tag{2.16}$$

where $F(\mathcal{P};q) = \sum_{x\in\mathcal{P}} q^{\rho(x)}$ is the rank-generating series of $\mathcal{P}$, a r-differential poset over a field $\mathbb{K}$ containing t.

2.4.2 Solving the Functional Equation

We now turn to the problem of finding a solution to the functional equation (2.15). First, we check that such a solution is unique. Writing $G(q,t) = \sum_{k\geq 0} g_k(q)t^k$ and $\exp(rt + \frac{rt^2}{2}) = \sum_{i\geq 0} a_i t^i$, the functions $g_k(q)$ satisfy

$$\begin{aligned} g_k(q) &= [t^k] \exp\left(rt + \frac{rt^2}{2}\right) G(q,qt) \\ &= \sum_{i\geq 0}\sum_{j\geq 0} a_i g_j(q) q^j t^{i+j} \\ &= \sum_{0\leq i\leq k} a_i q^{k-i} g_{k-i}(q). \end{aligned}$$

From this we obtain the linear recurrence

$$g_k(q) = \frac{1}{1-q^k} \sum_{1\leq i\leq k} a_i q^{k-i} g_{k-i}(q),$$

with initial condition $g_0(q) = F(\mathcal{P};q)$. Thus, the sequence $\{g_k\}_{k\geq 0}$ and $G(q,t)$ are uniquely determined.

To find this unique solution, note that equation (2.15) can be written as

$$\frac{G(q,t)}{G(q,qt)} = \exp\left(rt + \frac{1}{2}rt^2\right),$$

so, replacing t with $q^i t$,

$$\frac{G(q,q^i t)}{G(q,q^{i+1}t)} = \exp\left(rq^i t + \frac{1}{2}rq^{2i}t^2\right).$$

So, for any $N \geq 0$,

$$G(q,t) = G(q, q^{N+1}t) \prod_{0 \leq i \leq N} \exp\left(rq^i t + \frac{1}{2} r q^{2i} t^2\right).$$

Taking the limit as $N \to \infty$,

$$\begin{aligned} G(q,t) &= \lim_{N\to\infty} G(q, q^{N+1}t) \prod_{i \geq 0} \exp\left(rq^i t + \frac{1}{2} r q^{2i} t^2\right) \\ &= \lim_{N\to\infty} G(q, q^{N+1}t) \exp\left(\sum_{i\geq 0} rq^i t + \frac{1}{2} r q^{2i} t^2\right) \\ &= \lim_{N\to\infty} G(q, q^{N+1}t) \exp\left(\frac{rt}{1-q} + \frac{rt^2}{2(1-q^2)}\right). \end{aligned}$$

Writing $G(q,t) = \sum_{n\geq 0} a_n(q) t^n$,

$$\begin{aligned} \lim_{N\to\infty} G(q, q^{N+1}t) &= \sum_{n\geq 0} \lim_{N\to\infty} a_n(q) q^{N+1} t^n \\ &= a_0(q) \\ &= G(q,0) \\ &= F(\mathcal{P}; q). \end{aligned}$$

Thus, we have proven the following.

Theorem 2.27 (Stanley). *Let $\mathcal{P}$ be an r-differential poset over a field $\mathbb{K}$ containing t, and let $\alpha(n \to n+k)$ denote the number of paths of shape U^k from the n^{th} rank to the $(n+k)^{th}$ rank. Let*

$$G(q,t) := \sum_{n\geq 0} \sum_{k \geq 0} \alpha(n \to n+k) q^n \frac{t^k}{k!}.$$

Then

$$G(q,t) = F(\mathcal{P}; q) \exp\left(\frac{rt}{1-q} + \frac{rt^2}{2(1-q^2)}\right),$$

where $F(\mathcal{P}; q)$ is the rank-generating series of the poset $\mathcal{P}$.

Using Lemma 2.24, we can obtain the following formula.

	$n=0$	1	2	3	4	5
$k=0$	1	1	2	3	5	7
1	1	2	4	7	12	19
2	2	4	10	18	34	56
3	4	10	26	52	102	180
4	10	26	76	158	332	608
5	26	76	232	518	1130	2178

Table 2.3: Values of $\alpha(n \to n+k)$

Corollary 2.28.

$$\begin{aligned}\alpha(n \to n+k) &= [q^n]F(\mathcal{P};q) \sum_{\sigma \in \mathrm{inv}(n)} \left(\frac{1}{1-q}\right)^{2c(\sigma)-n} \left(\frac{1}{1-q^2}\right)^{n-c(\sigma)} \\ &= [q^n]F(\mathcal{P},q) \sum_{\sigma \in \mathrm{inv}(n)} \left(\frac{1}{1-q}\right)^{c(\sigma)} \left(\frac{1}{1+q}\right)^{n-c(\sigma)}.\end{aligned}$$

Example 2.29. Consider applying Theorem 2.27 to Young's Lattice. In this case, $r=1$ and the rank-generating series is just the generating series which counts the number of partitions of n, namely

$$F(\mathcal{Y};q) = \prod_{i \geq 0}(1-q^i)^{-1}.$$

So in this case, we have

$$G(q,t) = \prod_{i \geq 0}(1-q^i)^{-1} \exp\left(\frac{t}{1-q} + \frac{t^2}{2(1-q^2)}\right).$$

Extracting the coefficient of $q^n \frac{t^k}{k!}$ from this series will give $\alpha(n \to n+k)$, which in the case of Young's Lattice, is the number of skew-tableau having shape of the form $\lambda \setminus \mu$, where $\lambda \vdash n+k$ and $\mu \vdash n$. We can use Maple to extract coefficients from this series — several values are given in Table 2.3. For example, we should have $\alpha(2 \to 5) = 26$. Thirteen of the skew tableaux corresponding to these paths are shown in Figure 2.2 — the remaining 13 may be obtained by taking the transpose of each of these diagrams.

As a further specialization, set $q=0$ to get

$$\sum_{k \geq 0} \alpha(0 \to k)\frac{t^k}{k!} = \exp\left(t + \frac{t^2}{2}\right).$$

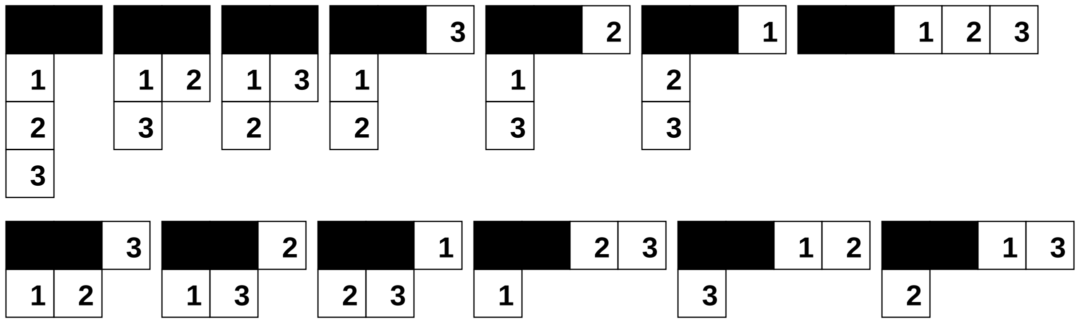


Figure 2.2: Half of the skew-tableaux corresponding to monotonic paths connecting rank 2 to rank 5 in Young's Lattice

This is the exponential generating series for the number of involutions in $\mathfrak{S}_k$, so it follows that the number of Standard Young Tableau with k boxes, *i.e.* $\alpha(0 \to k)$ is equal to the number of involutions in $\mathfrak{S}_k$.

More generally, for an r-differential poset, we have

$$\sum_{k \geq 0} \alpha(0 \to k) \frac{t^k}{k!} = \exp\left(rt + \frac{rt^2}{2}\right).$$

(Note that $F(0; \mathcal{P}) = 1$ for a differential poset, since $\hat{0}$ is the unique element of rank 0.) Then

$$\alpha(0 \to k) = \sum_{\sigma \in \mathrm{inv}(k)} r^{c(\sigma)}.$$

2.5 Enumerating Walks of a Single Fixed Shape

With the values $\alpha(n \to n+k)$ in hand, we are now in a position to compute $\alpha(n \xrightarrow{W} n+k)$.

Example 2.30. In Young's Lattice, continuing Example 2.21 with the numbers given in Table 2.3, we obtain

$$\begin{aligned}
\alpha(2 \xrightarrow{UUDDU} 3) &= 4\alpha(0 \to 3) + 2\alpha(1 \to 3) \\
&= 16 + 8 \\
&= 24.
\end{aligned}$$

(11, 111, 11, 1, 11,111)	(2, 3, 2, 1, 2, 3)
(11, 111, 11, 1, 11, 21)	(2, 3, 2, 1, 2, 21)
(11, 111, 11, 1, 2, 3)	(2, 3, 2, 1, 11, 111)
(11, 111, 11, 1, 2, 21)	(2, 3, 2, 1, 11, 21)
(11, 21, 11, 1, 11, 111)	(2, 21, 2, 1, 2, 3)
(11, 21, 11, 1, 11, 21)	(2, 21, 2, 1, 2, 21)
(11, 21, 11, 1, 2, 3)	(2, 21, 2, 1, 11, 111)
(11, 21, 11, 1, 2, 21)	(2, 21, 2, 1, 11, 21)
(11, 21, 2, 1, 11, 111)	(2, 21, 11, 1, 2, 3)
(11, 21, 2, 1, 11, 21)	(2, 21, 11, 1, 2, 21)
(11, 21, 2, 1, 2, 3)	(2, 21, 11, 1, 11, 111)
(11, 21, 2, 1, 2, 21)	(2, 21, 11, 1, 11, 21)

Table 2.4: The 24 walks in Young's Lattice of shape $UUDDU$, starting at the second rank

This result may be verified by constructing all walks of shape $UUDDU$ in Young's Lattice, starting at the second rank. These 24 walks are given in Table 2.4.

In general, it is possible to find the generating series for $\alpha(n \xrightarrow{W} n+k)$ for any W. Recall that, by Lemma 2.22, writing $W\mathbf{P} = p_W(U)\mathbf{P}$ for the unique polynomial $p_W(z) = \sum_{i\geq 0} a_i z^i$,

$$\alpha(n \xrightarrow{W} n+k) = \sum_{i\geq 0} a_i \alpha(n+k-i \to n+k).$$

Recall also that $\alpha(n+k-i \to n+k)$ is the coefficient of $q^{n+k-i}\frac{t^i}{i!}$ in the generating series $G(q,t)$ of Section 2.4. Thus

$$\alpha(n \xrightarrow{W} n+k) = \sum_{i\geq 0} a_i \left[q^{n+k-i}\frac{t^i}{i!}\right] G(q,t) = [q^{n+k}] \sum_{i\geq 0} a_i q^i \left[\frac{t^i}{i!}\right] G(q,t).$$

By Theorem 2.27, we know that $G(q,t) = F(\mathcal{P};q)\exp\left(\frac{rt}{1-q} + \frac{rt^2}{2(1-q^2)}\right)$. Since $F(\mathcal{P};q)$ is independent of t, then we have

$$\alpha(n \xrightarrow{W} n+k) = [q^{n+k}]F(\mathcal{P};q)\sum_{i\geq 0} a_i q^i \left[\frac{t^i}{i!}\right] \exp\left(\frac{rt}{1-q} + \frac{rt^2}{2(1-q^2)}\right).$$

Define a linear transformation $\Gamma : \mathbb{Q}[z] \to \mathbb{Q}[[q]]$ by

$$\Gamma(z^i) = q^i \left[\frac{t^i}{i!}\right] \exp\left(\frac{rt}{1-q} + \frac{rt^2}{2(1-q^2)}\right).$$

Then, we have the following result.

Theorem 2.31. *Let $\mathcal{P}$ be a r-differential poset over a field $\mathbb{K}$ containing t, $F(\mathcal{P};q)$ be its rank-generating series, and $W \in \{U,D\}^*$ be a word of displacement k. Let $p_W(z)$ be such that $W\mathbf{P} = p_W(U)\mathbf{P}$. Then*

$$\alpha(n \xrightarrow{W} n+k) = [q^{n+k}]F(\mathcal{P};q)\Gamma(p_W(z)).$$

The functions $\Gamma(z^i)$ may be computed by using Maple to extract coefficients from $\exp\left(\frac{rt}{1-q} + \frac{rt^2}{2(1-q^2)}\right)$, or by using the formula

$$\Gamma(z^i) = q^i \sum_{\sigma \in \mathrm{inv}(i)} \left(\frac{1}{1-q}\right)^{c(\sigma)} \left(\frac{1}{1+q}\right)^{n-c(\sigma)},$$

which we obtain by using Lemma 2.24. The first four functions are

$$\begin{aligned}
\Gamma(z^0) &= 1,\\
\Gamma(z) &= \frac{qr}{1-q},\\
\Gamma(z^2) &= q^2\frac{qr(r-1)+r(r+1)}{(1-q)^2(q+1)},\\
\Gamma(z^3) &= q^3\frac{qr^2(r-3)+r^2(r+3)}{(1-q)^3(1+q)}.
\end{aligned}$$

Example 2.32. Consider walks of the shape $W = DDU$. The polynomial corresponding to this word is

$$p_{DDU}(z) = z^3 + 2rz^2 + (r^2+3r)z + 2r.$$

So, applying Theorem 2.31,

$$\begin{aligned}
\alpha(n \xrightarrow{DDU} n-1) &= [q^{n-1}]F(\mathcal{P};q)\left(\Gamma(z^3) + 2r\Gamma(z^2) + (r^2+3r)\Gamma(z) + 2r\Gamma(1)\right)\\
&= [q^{n-1}]F(\mathcal{P};q)\frac{q^2r^2(r-1)+qr^2(r-1)+2r^2}{(1-q)^3(1+q)}.
\end{aligned}$$

In the case of Young's Lattice, set $r = 1$ and $F(\mathcal{P}; q) = \prod_{i\geq 0}(1 - q^i)^{-1}$. Expanding the series, we obtain

$$F(\mathcal{P}; q)\frac{2}{(1-q)^3(1+q)} = 2 + 6q + 16q^2 + 34q^3 + 68q^4 + 124q^5 + \cdots .$$

We can then read off various values of $\alpha(n \xrightarrow{DDU} n-1)$ as follows.

$$\begin{aligned}
\alpha(1 &\xrightarrow{DDU} 0) = 2,\\
\alpha(2 &\xrightarrow{DDU} 1) = 6,\\
\alpha(3 &\xrightarrow{DDU} 2) = 16,\\
\alpha(4 &\xrightarrow{DDU} 3) = 34,\\
\alpha(5 &\xrightarrow{DDU} 4) = 68,\\
\alpha(6 &\xrightarrow{DDU} 5) = 124.
\end{aligned}$$

It can be verified that these numbers are correct by listing walks on Young's Lattice — for example, the 6 walks of shape DDU starting at rank 2 are $(11, 111, 11, 1), (11, 21, 11, 1), (11, 21, 2, 1), (2, 3, 2, 1), (2, 21, 2, 1)$ and $(2, 21, 11, 1)$.

2.5.1 Walks of Shape D^kU^k

This section deals with the application of the techniques of the preceding section to the problem of enumerating walks of the shape D^kU^k. This will provide a generalization of equation (1.1) to the more general setting of r-differential posets. Furthermore, in Young's Lattice, the formula can also be generalized to a sum over skew tableaux.

The first task is to compute the polynomial $p_{D^kU^k}(z)$. Using Corollary 2.14,

$$\begin{aligned}
p_{D^kU^k}(U)\mathbf{P} &= D^kU^k\mathbf{P}\\
&= \sum_{0\leq i\leq k}\binom{k}{i}r^i\frac{\partial^i}{\partial U^i}(U^k)D^{k-i}\mathbf{P}\\
&= \sum_{0\leq i\leq k}\binom{k}{i}^2 i!r^iU^{k-i}p_{k-i}(U)\mathbf{P}.
\end{aligned}$$

By the uniqueness result of Theorem 2.16,

$$p_{D^kU^k}(z) = \sum_{0\le i\le k} \binom{k}{i}^2 i! r^i z^{k-i} p_{k-i}(z).$$

By Theorem 2.31,

$$\alpha(n \xrightarrow{D^kU^k} n) = [q^n]F(\mathcal{P};q) \sum_{0\le i\le k} \binom{k}{i}^2 i! r^i \Gamma(z^{k-i} p_{k-i}(z)).$$

The easiest of these values to compute is $\alpha(0 \xrightarrow{D^kU^k} 0)$, since it may be computed by setting $q = 0$. Since $F(0;\mathcal{P}) = 1$ for any differential poset $\mathcal{P}$, $\alpha(0 \xrightarrow{D^kU^k} 0)$ will depend only on r. Since $\Gamma(t^{k-i} p_{k-i}(z))$ is divisible by q^j whenever $k - i \ge j$, then the only nonzero term in the above sum occurs when $i = k$. Thus

$$\alpha(0 \xrightarrow{D^kU^k} 0) = \binom{k}{k}^2 k! r^k \Gamma(p_0(z)) = r^k k! \Gamma(1) = r^k k!.$$

In Young's Lattice, set $r = 1$ to obtain the familiar result of equation (1.1).

Values of $\alpha(n \xrightarrow{D^kU^k} n)$ for $n > 0$ will depend on the rank-generating series of the poset. These values can be computed using the polynomials $p_k(z)$ and the action of Γ.

2.6 Closed Walks

This section deals with the enumeration of **closed walks**, that is, walks that start and end at the same point. The strategy parallels that used to enumerate non-closed walks, as discussed in the previous sections, though there are some additional difficulties which arise. In particular, given a monomial W, we will be dealing with the numbers

$$\kappa(n, W) := \sum_{x\in\mathcal{P}_n} \langle x, Wx\rangle,$$

which cannot be written in terms of the vector $\mathbf{P}$. Thus, we will not be able to rely on key results such as Theorem 2.16 and its consequences.

The strategy used to calculate $\alpha(n \xrightarrow{W} n + k)$ in the three preceding sections is as follows.

1. Write W in some canonical form. (That is, writing $W\mathbf{P}$ in terms of a polynomial in U, using Theorem 2.16.)

2. Use the canonical form to express $\alpha(n \xrightarrow{W} n+k)$ in terms of simpler numbers, that is, in terms of $\alpha(n \to k)$, as in Lemma 2.22.

3. Compute the generating series of the simpler numbers (as in Theorem 2.27).

4. Use this generating series to find the generating series for $\alpha(n \xrightarrow{W} n+k)$ (as in Theorem 2.31).

The process of computing $\kappa(n, W)$ begins with writing W in a canonical form, though we must use a canonical form that does not rely on Theorem 2.16. In dealing with closed walks, we need only consider words in which the number of U's and the number of D's are the same. Such a word is called **balanced**. In general, a polynomial is called balanced if it is a linear combination of balanced monomials. Since the equation

$$Dg(U) = r\frac{\partial}{\partial U}g(U) + g(U)D$$

of Theorem 2.12 still applies, a reasonable strategy is to use this equation to move all instances of D to the right, resulting in a linear combination of $U^k D^k$, as shown in the following.

Example 2.33.

$$\begin{aligned} DUDU &= DU(UD + r) \\ &= DUUD + rDU \\ &= (2rU + U^2D)D + r(UD + r) \\ &= U^2D^2 + 3rUD + r^2. \end{aligned}$$

Let Δ denote the linear transformation $\Delta : \mathbb{K}[z] \to \mathrm{End}(\mathbb{K}^{\mathcal{P}})$ given by

$$\Delta(z^i) = U^i D^i.$$

Our objective is now to prove the following theorem.

Theorem 2.34. *Let $W \in \{U, D\}^*$ be a balanced word. Then there exists a unique polynomial $b_W \in \mathbb{K}[z]$ such that*

$$W = \Delta(b_W(z)).$$

Moreover, b_W is given recursively by

$$b_{U^k D^k} = z^k \text{ for all } k,$$
$$b_{V_1 DUV_2} = b_{V_1 UDV_2} + r b_{V_1 V_2},$$

where $V_2 = U^i D^j$ for some $i, j \geq 0$.

Proof: Define the "badness" of a word, $\beta(W)$, as follows. If $W = A_1 A_2 \ldots A_n$ where $A_i \in \{U, D\}$, $\beta(W)$ is the number of pairs $i < j$ such that $A_i = D$ and $A_j = U$. For example, $\beta(DUDU) = 3$ since the first D is to the left of 2 U's, and the second D is to the left of only one U. Proceed by induction on $\beta(W)$.

The base case is $\beta(W) = 0$. In this case, W is of the form $U^k D^k$ for some k. Define $b_{U^k D^k} = z^k$ and note that $\Delta(b_{U^k D^k}) = \Delta(z^k) = U^k D^k$, so this is indeed the desired polynomial.

Suppose that $\beta(W) > 0$ and that the result holds for all words V with $\beta(V) < \beta(W)$. Since $\beta(W) \neq 0$, then W is of the form $V_1 DUV_2$ where $V_2 = U^i D^j$ for some $i, j \geq 0$. Note that $\beta(W) = \beta(V_1 V_2) + \sharp_U(V_2) + \sharp_D(V_1) + 1$, where $\sharp_U(V)$ (respectively, $\sharp_D(V)$), denotes the number of occurrences of U (resp. D) in the monomial V. Now,

$$W = V_1 DUV_2 = V_1(UD + r)V_2 = V_1 UDV_2 + rV_1 V_2.$$

Observe that $\beta(V_1 UDV_2) = \beta(V_1 V_2) + \sharp_U(V_2) + \sharp_D(V_1) < \beta(W)$ and that $\beta(V_1 V_2) < \beta(W)$. Thus, by induction, there exist polynomials $b_{V_1 UDV_2}$ and $b_{V_1 V_2}$ such that

$$V_1 UDV_2 = \Delta(b_{V_1 UDV_2})$$

and

$$V_1 V_2 = \Delta(b_{V_1 V_2}).$$

Thus

$$\begin{aligned} W &= V_1 UDV_2 + rV_1 V_2 \\ &= \Delta(b_{V_1 UDV_2}) + r\Delta(b_{V_1 V_2}) \\ &= \Delta(b_{V_1 UDV_2} + r b_{V_1 V_2}). \end{aligned}$$

Thus, by defining $b_W = b_{V_1UDV_2} + rb_{V_1V_2}$, the existence of such a polynomial is shown. As for uniqueness, if there are polynomials b_1 and b_2 such that $\Delta(b_1) = W = \Delta(b_2)$, then it is clear that $b_1 = b_2$ since Δ is injective. $\square$

Note that if $b_W = \sum_{i\geq 0} b_i z^i$, then

$$\begin{aligned}
\kappa(n, W) &= \sum_{x\in\mathcal{P}_n} \langle x, Wx\rangle \\
&= \sum_{x\in\mathcal{P}_n} \left\langle x, \sum_{i\geq 0} b_i \Delta(z^i) x \right\rangle \\
&= \sum_{i\geq 0} b_i \sum_{x\in\mathcal{P}_n} \langle x, U^i D^i x\rangle \\
&= \sum_{i\geq 0} b_i \kappa(n, U^i D^i),
\end{aligned}$$

so we have the following result.

Corollary 2.35. *Let $W \in \{U, D\}^*$, and suppose $b_W = \sum_{i\geq 0} b_i z^i$. Then*

$$\kappa(n, W) = \sum_{i\geq 0} b_i \kappa(n, U^i D^i).$$

Our task now is to compute the values $\kappa(n, U^k D^k)$. The following lemma will be helpful.

Lemma 2.36.

$$\kappa(n, U^k D^k) = \kappa(n - k, D^k U^k).$$

Proof: If $k > n$, then both values are 0, so we need only consider the case when $n \geq k$. It should be clear from a combinatorial point of view that these numbers are equal, since they both count closed walks connecting the n^{th} rank and the $(n-k)^{\text{th}}$ rank.

For an algebraic proof, note that if $x \in \mathcal{P}_n$, then

$$D^k x = \sum_{y\in\mathcal{P}_{n-k}} \left\langle y, D^k x \right\rangle y,$$

and that for any $y \in \mathcal{P}_{n-k}$,

$$U^k y = \sum_{x \in \mathcal{P}_n} \left\langle x, U^k y \right\rangle x.$$

Using the fact that $U^* = D$ and that $\langle \cdot, \cdot \rangle$ is symmetric on arguments of finite support,

$$\begin{aligned}
\kappa(n, U^k D^k) &= \sum_{x \in \mathcal{P}_n} \left\langle x, U^k D^k x \right\rangle \\
&= \sum_{x \in \mathcal{P}_n} \sum_{y \in \mathcal{P}_{n-k}} \left\langle y, D^k x \right\rangle \left\langle x, U^k y \right\rangle \\
&= \sum_{y \in \mathcal{P}_{n-k}} \sum_{x \in \mathcal{P}_n} \left\langle x, U^k y \right\rangle \left\langle x, U^k y \right\rangle \\
&= \sum_{y \in \mathcal{P}_{n-k}} \left\langle U^k y, U^k y \right\rangle \\
&= \sum_{y \in \mathcal{P}_{n-k}} \left\langle y, D^k U^k y \right\rangle \\
&= \kappa(n-k, D^k U^k).
\end{aligned}$$

□

We are now in a position to state a recursive formula for $\kappa(n, U^k D^k)$. By Corollary 2.14,

$$D^k U^k = \sum_{0 \le i \le k} \binom{k}{i}^2 i! r^i U^{k-i} D^{k-i}.$$

Combining this with Corollary 2.35 and Lemma 2.36, we obtain the formula

$$\kappa(n, U^k D^k) = \sum_{0 \le i \le k} \binom{k}{i}^2 i! r^i \kappa(n-k, U^{k-i} D^{k-i}). \tag{2.17}$$

In the same spirit as the proof of Lemma 2.26, we can derive a functional equation for the following generating series for $\kappa(n, U^k D^k)$.

$$\begin{aligned}
H(q,t) &:= \sum_{k\geq 0}\sum_{n\geq 0} \kappa(n, U^k D^k) q^n \frac{t^k}{(k!)^2} \\
&= \sum_{k\geq 0}\sum_{n\geq 0}\sum_{0\leq i\leq k} \binom{k}{i}^2 i! r^i \kappa(n-k, U^{k-i} D^{k-i}) q^n \frac{t^k}{(k!)^2} \\
&= \sum_{k\geq 0}\sum_{n\geq 0}\sum_{0\leq i\leq k} \frac{r^i}{i!(k-i)!^2} \kappa(n-k, U^{k-i} D^{k-i}) q^n t^k.
\end{aligned}$$

Changing the order of summation and reindexing to increment k by i,

$$\begin{aligned}
H(q,t) &= \sum_{n\geq 0}\sum_{i\geq 0}\sum_{k\geq 0} \frac{r^i}{i!} \kappa(n-k-i, U^k D^k) q^n \frac{t^{k+i}}{(k!)^2} \\
&= \sum_{n\geq 0}\sum_{i\geq 0}\sum_{k\geq 0} \kappa(n-k-i, U^k D^k) q^{n-k-i} \frac{(qt)^k}{(k!)^2} \frac{(rqt)^i}{i!} \\
&= \exp(rqt) H(q, qt).
\end{aligned}$$

The initial condition for this functional equation is

$$H(q,0) = \sum_{n\geq 0} \kappa(n, I) q^n = F(\mathcal{P}; q),$$

where $F(\mathcal{P}; q)$ is the rank-generating series of $\mathcal{P}$. We can find the unique solution to this functional equation using the technique of Section 2.4.2. Note that for any N,

$$\frac{H(q, q^N t)}{H(q, q^{N+1} t)} = \exp(rq^{N+1} t),$$

so

$$H(q,t) = H(q, q^{N+1} t) \prod_{0\leq i\leq N} \exp(rq^{i+1} t).$$

Taking the limit as $N \to \infty$,

$$\begin{aligned}
H(q,t) &= H(q,0) \prod_{i\geq 0} \exp(rq^{i+1} t) \\
&= F(\mathcal{P}; q) \exp\left(\sum_{i\geq 0} rq^{i+1} t \right) \\
&= F(\mathcal{P}; q) \exp\left(\frac{rqt}{1-q} \right).
\end{aligned}$$

Thus, we have proven the following theorem.

	$n=0$	1	2	3	4	5	6	7
$k=0$	1	1	2	3	4	5	11	15
1	0	1	2	4	7	12	19	30
2	0	0	2	6	14	28	52	90
3	0	0	0	6	24	66	150	306
4	0	0	0	0	24	120	384	984

Table 2.5: Values of $\kappa(n, U^k D^k)$ for Young's Lattice

Theorem 2.37 (Stanley). *If $\mathcal{P}$ is a r-differential poset over a field $\mathbb{K}$ containing t, and*

$$H(q,t) = \sum_{k\geq 0}\sum_{n\geq 0} \kappa(n, U^k D^k) q^n \frac{t^k}{(k!)^2},$$

then

$$H(q,t) = F(\mathcal{P}; q) \exp\left(\frac{rqt}{1-q}\right),$$

where $F(\mathcal{P}; q)$ is the rank-generating series for $\mathcal{P}$.

This theorem has the following immediate corollary.

Corollary 2.38.

$$\kappa(n, U^k D^k) = k! r^k [q^n] F(\mathcal{P}; q) \frac{q^k}{(1-q)^k}.$$

Some values of $\kappa(n, U^k D^k)$ for Young's Lattice, computed using Maple, are given in Table 2.5.

We are now in a position to state a formula for $\kappa(n, W)$.

Theorem 2.39 (Stanley). *Let $\mathcal{P}$ be a r-differential poset over a field $\mathbb{K}$ containing t, and let $W \in \{U, D\}^*$. Let $b_W = \sum_{i\geq 0} b_i z^i$ be such that $W = \Delta(b_W)$. Then*

$$\kappa(n, W) = [q^n] F(\mathcal{P}; q) \sum_{i\geq 0} b_i \frac{i!(rq)^i}{(1-q)^i},$$

where $F(\mathcal{P}; q)$ is the rank-generating series of $\mathcal{P}$.

11,111,11,111,11	2,3,2,3,2
11,111,11,21,11	2,3,2,21,2
11,21,11,21,11	2, 21,2, 21, 2
11,21,11,111,11	2,21,2,3,2
11,21,2,21,11	2, 21, 11,21,2

Table 2.6: The ten closed walks counted by $\kappa(2, DUDU)$

Proof: Combining Corollaries 2.35 and 2.38,

$$\begin{aligned}\kappa(n, W) &= \sum_{i\geq 0} b_i \kappa(n, U^i D^i)\\ &= \sum_{i\geq 0} b_i i! r^i [q^n] F(\mathcal{P}; q) \frac{q^i}{(1-q)^i}\\ &= [q^n] F(\mathcal{P}; q) \sum_{i\geq 0} b_i \frac{i!(rq)^i}{(1-q)^i}.\end{aligned}$$

□

Example 2.40. We can now continue Example 2.33. We have computed

$$DUDU = U^2D^2 + 3rUD + r^2.$$

Applying Theorem 2.39,

$$\kappa(n, DUDU) = [q^n]F(\mathcal{P}; q)\left(\frac{2r^2q^2}{(1-q)^2} + \frac{3r^2q}{1-q} + r^2\right).$$

In the case of Young's Lattice, this gives the generating series

$$F(\mathcal{Y}; q)\left(\frac{2q^2}{(1-q)^2} + \frac{3q}{1-q} + 1\right) = 1+4q+10q^2+21q^3+40q^4+71q^5+120q^6+\cdots .$$

We can verify this by counting closed walks in Young's Lattice. For example, the $\kappa(2, DUDU) = 10$ closed walks are given in Table 2.6.

2.7 Exponential Functions of U and D

While the previous section dealt with the problem of computing the number of walks of a fixed shape, it is also possible to develop machinery for dealing with the problem of enumerating walks whose shape can be any element of a suitably described set of shapes — for example, "all walks of length k." It is clear that the polynomial $(U+D)^k$ is the endomorphism of $\mathbb{K}^{\mathcal{P}}$ corresponding to walks of length k. Thus, one expects that it will be useful to be able to deal with expressions such as

$$\sum_{k\geq 0}(U+D)^k\frac{t^k}{k!} = \exp((U+D)t).$$

In order to deal with exponential functions of U and D, we shall develop a partial differential equation whose solution is a formal power series in U, which, when applied to $\mathbf{P}$, is equal to

$$\exp((f(U)+g(D))t)h(U)\mathbf{P},$$

where $f, h \in \mathbb{K}[[U]]$ and $g = \sum_{1\leq i\leq n} g_i D^i \in \mathbb{K}[D]$ with $g(0) \neq 0$. By Corollary 2.17, there is a unique formal power series $H(U,t)$ such that

$$\exp((f(U)+g(D))t)h(U)\mathbf{P} = H(U,t)\mathbf{P}. \tag{2.18}$$

Differentiating with respect to t,

$$\begin{aligned}\frac{\partial H}{\partial t}\mathbf{P} &= (f(U)+g(D))\exp((f(U)+g(D))t)h(U)\mathbf{P}\\ &= (f(U)H(U,t)+g(D)H(U,t))\mathbf{P}.\end{aligned}$$

Applying Corollary 2.14,

$$\begin{aligned}\frac{\partial H}{\partial t}\mathbf{P} &= \left(f(U)H(U,t)+\sum_{1\leq i\leq n} g_i \sum_{0\leq j\leq i}\binom{i}{j} r^j \frac{\partial^j H}{\partial U^j} D_{i-j}\right)\mathbf{P}\\ &= \left(f(U)H(U,t)+\sum_{1\leq i\leq n} g_i \sum_{0\leq j\leq i}\binom{i}{j} r^j \frac{\partial^j H}{\partial U^j} p_{i-j}(U)\right)\mathbf{P},\end{aligned}$$

where the polynomials $p_k(z)$ are those from Section 2.4. Applying the same argument as the uniqueness part of Theorem 2.16, we have proven the following.

Lemma 2.41. *Let $\mathcal{P}$ be a r-differential poset over a field $\mathbb{K}$ containing t. Let $f, h \in \mathbb{K}[[U]]$ and $g = \sum_{1\leq i\leq n} g_i D^i$. Let $H(U,t)$ be such that*

$$\exp((f(U)+g(D))t)h(U)\mathbf{P} = H(U,t)\mathbf{P}.$$

Then $H(U,t)$ is a solution to the partial differential equation

$$\frac{\partial H}{\partial t} = fH + \sum_{1\le i\le n} g_i \sum_{0\le j\le i} \binom{i}{j} r^j p_{i-j}(U) \frac{\partial^j H}{\partial U^j},$$

where $p_k(z) = k![t^k]\exp(rt + zt + \frac{t^2 r}{2})$.

Note that this partial differential equation is of order 1 in t and order $\deg(g)$ in U. In general, it is also non-linear, and we are interested in a formal solution, so the problem of solving this equation is extremely difficult. In [17], Stanley provides a formula which solves this partial differential equation in the case $\deg(g) = 1$. Given the difficulty of solving this equation in general, it seems that Stanley's theorem is the most general statement we shall be able to make about exponential functions of U and D.

Theorem 2.42 (Stanley). *Let $\mathcal{P}$ be a r-differential poset over a field $\mathbb{K}$ containing t. Let $c \in \mathbb{K}$ and $f, h \in \mathbb{K}[[U]]$. Then*

$$\exp(f(U)+cD)t)h(U)\mathbf{P} = \exp\left(crt + \frac{1}{2}c^2rt^2 + cUt + \int_0^t f(U+crs)ds\right) h(U+crt)\mathbf{P}.$$

Proof: Taking $g(D) = cD$ in Lemma 2.41, we obtain the partial differential equation

$$\begin{aligned}\frac{\partial H}{\partial t} &= fH + c(p_1(U)H + rp_0(U)\frac{\partial H}{\partial U}) \\ &= (f + cU + cr)H + cr\frac{\partial H}{\partial U}.\end{aligned}$$

The initial condition for this partial differential equation can be found by setting $t = 0$ in equation (2.18). We obtain $h(U) = H(U, 0)$. This partial differential equation may be solved by applying the method of characteristics, which is described in Appendix B. This method reduces the problem to the following system of ordinary differential equations.

$$\frac{dU}{dt} = -cr; \tag{2.19}$$

$$\frac{dH}{dt} = (f(U) + cU + cr)H. \tag{2.20}$$

Equation 2.19 may be solved trivially by integration, to obtain

$$U = -crt + U_0$$

for some constant of integration U_0, independent of t. Note that with this additional relationship, the initial condition may be written as $H(U, 0) = h(U_0)$.

To solve equation (2.20), divide by H, substitute $U = U_0 - crt$ and integrate to obtain

$$\int_0^t \frac{dH}{H} = \int_0^t (f(U_0 - crs) + cU_0 - c^2rs + cr)ds,$$

so

$$\log(H(U,t)) + \log(H(U,0)) = \int_0^t f(U_0 - crs)ds + cU_0t - \frac{1}{2}c^2rt^2 + crt.$$

We may rewrite the integral in this expression by making the substitution $U_0 - crs = U + cr\sigma$ and noting that when $s = 0, \sigma = t$ and when $s = t, \sigma = 0$. Applying the change of variables theorem for integration,

$$\begin{aligned}\int_0^t f(U_0 - crs)ds &= -\int_t^0 f(U + cr\sigma)d\sigma \\ &= \int_0^t f(U + crs)ds.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\log(H) &= \int_0^t f(U + crs)ds + c(U + crt)t - \frac{1}{2}c^2rt^2 + crt + \log(h(U)) \\ &= \int_0^t f(U + crs)ds + cUt + crt + \frac{1}{2}c^2rt^2 + \log(h(U_0)).\end{aligned}$$

Hence,

$$H(U,t) = \exp\left(+cUt + crt + \frac{1}{2}c^2rt^2 + \int_0^t f(U + crs)ds\right) h(U + crt).$$

$\square$

Note that this is a generalization of Theorem 2.23. Setting $f(U) = U$, $h(U) = 1$ and $c = 1$, we obtain

$$\sum_{k\geq 0} p_k(U)\mathbf{P}\frac{t^k}{k!} = \sum_{k\geq 0} D^k\mathbf{P}\frac{t^k}{k!} = \exp(Dt)\mathbf{P} = \exp\left(rt + \frac{rt^2}{2} + Ut\right)\mathbf{P}.$$

2.7.1 Applications

The following example illustrates a way in which this theorem may be applied.

Example 2.43. Consider walks in which the maximal decreasing subwalks have length divisible by k. Let $\omega_{n,k,l}$ denote the number of walks of length $l+(n-l)k$, with a total of l "up" steps, starting at $\hat{0}$. By marking instances of U with the indeterminate z, the coefficient of z^l in $(zU + D^k)^n$ is the endomorphism corresponding to this set of shapes. Thus, the number we are interested in is

$$\begin{aligned}\omega_{n,k,l} &= [z^l]\langle (zU + D^k)^n \hat{0}, \mathbf{P}\rangle \\ &= [z^l]\langle \hat{0}, (U^k + zD)^n \mathbf{P}\rangle.\end{aligned}$$

Thus, the exponential generating series for $\omega_{n,k,l}$ is

$$\begin{aligned}D(z,t) &:= \sum_{n\geq 0}\sum_{l\geq 0} \omega_{n,k,l} z^l \frac{t^n}{n!} \\ &= \sum_{n\geq 0} \langle \hat{0}, (U^k + zD)^n \mathbf{P}\rangle \frac{t^n}{n!} \\ &= \langle \hat{0}, \exp((U^k + zD)t)\mathbf{P}\rangle.\end{aligned}$$

Setting $f(U) = U^k$, $h(U) = 1$ and $c = z$ in Theorem 2.42, we obtain

$$D(z,t) = \left\langle \hat{0}, \exp\left(zrt + \frac{1}{2}z^2rt^2 + zUt + \frac{1}{zr(k+1)}((U+zrt)^{k+1} - U^{k+1})\right)\mathbf{P}\right\rangle.$$

In order to evaluate this expression, we will use the following result.

Lemma 2.44 (Stanley). *Let $f \in \mathbb{K}[[U]]$. Then*

$$\left\langle \hat{0}, f(U)\mathbf{P}\right\rangle = f(0).$$

Proof: We can write $f(U) = f(0) + g(U)U$ for some $g \in \mathbb{K}[[U]]$. Note that $g(U)U\mathbf{P} \in \prod_{i\geq 1}\mathbb{K}^{\mathcal{P}_i}$, so

$$\begin{aligned}\left\langle \hat{0}, f(U)\mathbf{P}\right\rangle &= \left\langle \hat{0}, f(0)\mathbf{P}\right\rangle + \left\langle \hat{0}, g(U)U\mathbf{P}\right\rangle \\ &= f(0)\left\langle \hat{0}, \mathbf{P}\right\rangle \\ &= f(0),\end{aligned}$$

proving the lemma. $\square$

We may now continue Example 2.43 by setting $U = 0$, to obtain

$$D(z,t) = \exp\left(zrt + \frac{1}{2}z^2rt^2 + \frac{z^kr^kt^{k+1}}{k+1}\right).$$

Observe that these quantities depend only on r, and not on the rank-generating series of the differential poset. We can set $k = 1$ to obtain the generating series for the number $\omega_{n,l}$ of all walks of length n with l up-steps, namely

$$\exp\left(zrt + (1+z)\frac{zrt^2}{2}\right).$$

Note that for any n, the numbers $\omega_{n,l}$ are zero if $l > n$. Thus, it follows that $D(z,t)$ is a formal power series in t whose coefficients are polynomials in z. Thus, if we let $\omega_n = \sum_{l\geq 0}\omega_{n,l}$ be the total number of walks of length n, we may evaluate this at $z = 1$ to obtain

$$\sum_{n\geq 0}\omega_n\frac{t^n}{n!} = \exp(rt + rt^2),$$

which is a result of Stanley [17, Proposition 3.1]. We can apply Lemma 2.24 (with $a = r$ and $b = 2r$) to obtain the formula

$$\omega_n = \sum_{\sigma\in\mathrm{inv}(n)} r^{c(\sigma)}2^{n-c(\sigma)}.$$

Applying Lemma 2.24 to the generating series $\exp\left(zrt + (1+z)\frac{zrt^2}{2}\right)$, where $a = zr$ and $b = (1+z)zr$, we obtain the following formula for the coefficient of $\frac{t^n}{n!}$.

$$\sum_{l\geq 0}\omega_{n,l}z^l = \sum_{\sigma\in\mathrm{inv}(n)}(rz)^{c(\sigma)}(1+z)^{n-c(\sigma)}.$$

As a sample calculation, consider the case when $n = 3$. $\mathfrak{S}_3$ has four involutions, one of which (the identity) has 3 cycles, and three of which (namely, $(12), (13)$ and (23)) have 2 cycles. Thus

$$\sum_{l\geq 0}\omega_{3,l}z^l = (rz)^3 + 3(rz)^2(1+z) = 3r^2z^2 + (r^3+3r^2)z^3.$$

Chapter 3

Generalized Differential Posets

3.1 Motivation and Definitions

3.1.1 The Partial Order of Rooted Trees

In this chapter, the definition of a differential poset is extended so that it applies to a wider range of problems. An example, involving a partially ordered set of rooted trees, is introduced in this section as motivation. This example was studied by Hoffman [8], using methods similar to those in the previous chapter.

Let $\mathcal{T}$ denote the set of unlabelled rooted trees. Define a partial order on this set by $t \leq t'$ if and only if t can be obtained by deleting some set of non-root vertices of t'. Under this definition, it is clear that $t \lessdot t'$ if and only if t is obtained by deleting exactly one leaf vertex of t' (or, conversely, if t' may be obtained by adding an edge and a leaf vertex.) Thus, $\mathcal{T}$ is graded by $\rho(t) = |V(t)| - 1$, that is, the number of non-root vertices. The Hasse diagram of this poset is shown in Figure 3.1.

Ideally, we would like walks in this poset to correspond to the possible outcomes of adding or deleting vertices, so that enumeration of walks of a specified shape between two trees t and t' corresponds to the number of ways we can obtain t' from t subject to a specified sequence of vertex additions or

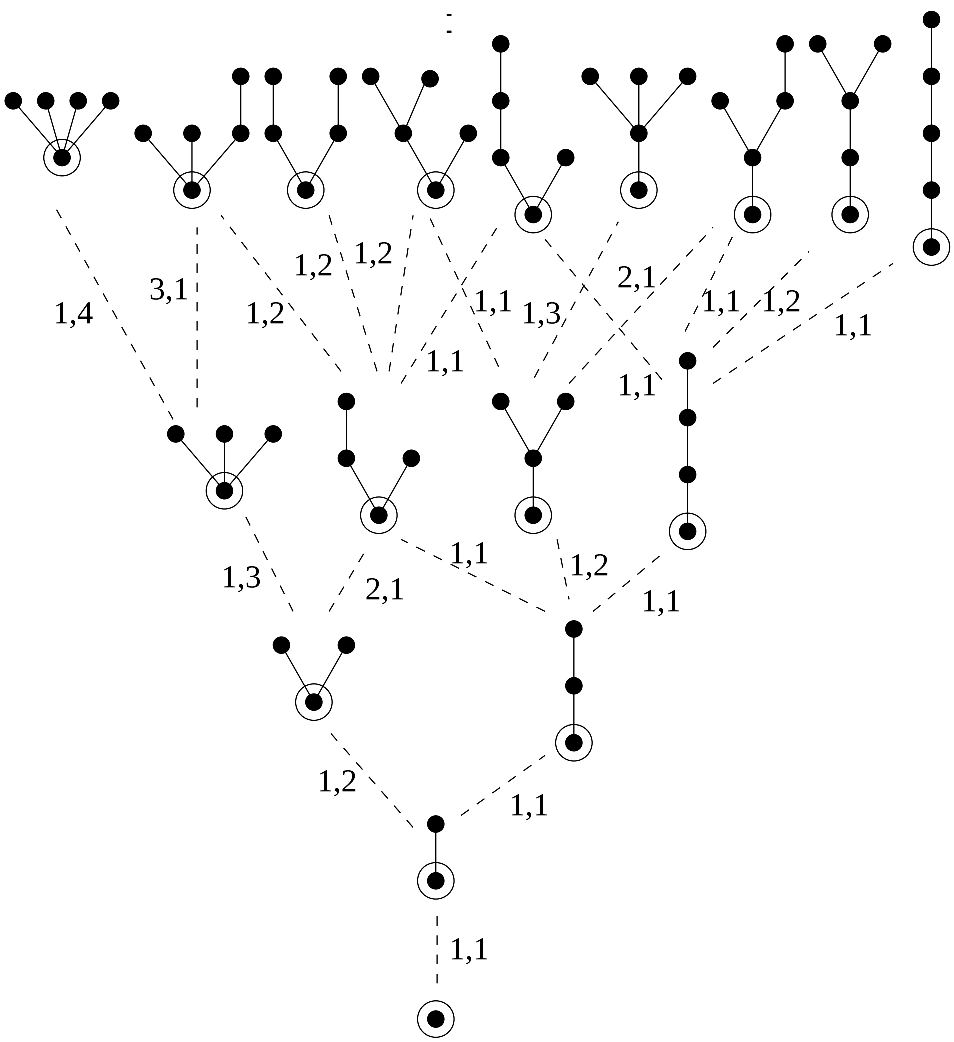


Figure 3.1: Partially ordered set of rooted trees, with weights $u(x,y), d(x,y)$ on cover relations

deletions. However, there is a slight problem with approaching this problem using the operators U and D given by

$$U(x) = \sum_{x \lessdot y} y = \sum_{y \in \mathcal{T}} c(x,y)y$$

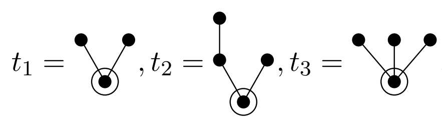
and

$$D(x) = \sum_{y \lessdot x} y = \sum_{y \in \mathcal{T}} c(y,x)y,$$

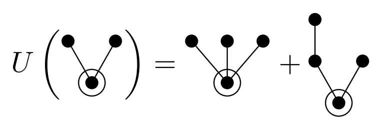
where $c(x,y)$ is a cover indicator function, given by

$$c(x,y) = \begin{cases} 1 & \text{if } x \lessdot y, \\ 0 & \text{otherwise.} \end{cases}$$

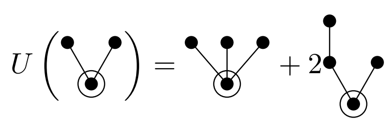
Using these same operators will not yield the desired result. To see this, let



Note that



However, this operation does not reflect the fact that there are two vertices of t_1 to which leaves may be affixed to give t_2 — informally, there are two ways in which t_2 covers t_1. In order to take this into account, it is necessary to have



The solution is to assign a weight to the cover relations of $\mathcal{T}$, and to use this weight in place of the cover indicator function. Define the function $u(x,y)$ to be the number of vertices of x to which affixing a leaf results in y. (Note that if y does not cover x, then $u(x,y) = 0$.) Then the operator

$$U(x) = \sum_{y \in \mathcal{T}} u(x,y)y$$

achieves the desired effect.

Replacing the cover indicator function with a more general weight function presents a further difficulty. Even though there are two ways to obtain

t_2 from t_1, there is only one leaf of t_2 which, when deleted, gives t_1. Similarly, even though there is only one place to add a leaf to t_1 to obtain t_3, there are three vertices of t_3 which, when deleted, give t_1. It is now clear that we cannot use the same cover weights to define D. Instead, let $d(x, y)$ be the number of vertices of y which, when deleted, give x. Then, define the operator D by

$$D(x) = \sum_{y \in \mathcal{T}} d(y, x) y.$$

The poset $\mathcal{T}$, together with its two cover weights, is shown in Figure 3.1.

The commutator of U and D can be computed using a combinatorial argument due to Hoffman. Let t be a rooted tree, with n vertices, k of which are leaves. Fix a labelling $\{v_i\}_{1 \leq i \leq n}$ of the vertices, so that $\{v_i\}_{1 \leq i \leq k}$ are the leaves of t. To obtain a tree which covers t in $\mathcal{T}$, a new leaf may be added to any of the n vertices of t. Denote the tree obtained by adding a new leaf v at the vertex v_i by $t \boxplus i$. Then, in this notation,

$$U(t) = \sum_{1 \leq i \leq n} t \boxplus i$$

and hence

$$DU(t) = \sum_{1 \leq i \leq n} D(t \boxplus i).$$

Now, consider the removal of leaf vertices from the trees $t \boxplus i$. There are two cases to consider. If $1 \leq i \leq k$, then the new vertex v was added to the leaf v_i of t. Thus, the leaves of $t \boxplus i$ are $\{v\} \cup \{v_j\}_{1 \leq j \leq k, i \neq j}$. So, in this case,

$$D(t \boxplus i) = (t \boxplus i) \setminus \{v\} + \sum_{1 \leq j \leq k, i \neq j} (t \boxplus i) \setminus \{v_j\} = t + \sum_{1 \leq j \leq k, i \neq j} (t \boxplus i) \setminus \{v_j\}.$$

In the remaining case, $k + 1 \leq i \leq n$, the new vertex v was not added to a leaf, so the leaves of $t \boxplus i$ are $\{v\} \cup \{v_j\}_{1 \leq j \leq k}$, in which case,

$$D(t \boxplus i) = t + \sum_{1 \leq j \leq k} (t \boxplus i) \setminus \{v_j\}.$$

Combining these two cases,

$$
\begin{aligned}
DU(t) &= \sum_{1\le i\le k} D(t \boxplus i) + \sum_{k+1\le i\le n} D(t\boxplus i) \\
&= \sum_{1\le i\le k}\left(t + \sum_{1\le j\le k, i\ne j} (t\boxplus i)\setminus\{v_j\}\right) + \sum_{k+1\le i\le n}\left(t + \sum_{1\le j\le k}(t\boxplus i)\setminus\{v_j\}\right) \\
&= nt + \sum_{1\le i\le n}\sum_{1\le j\le k, j\ne i} (t\boxplus i)\setminus\{j\}.
\end{aligned}
$$

Next, we must compute

$$UD(t) = \sum_{1\le j\le k} U(t\setminus\{v_j\}).$$

A new leaf vertex may be added at any of the vertices of $t\setminus\{v_j\}$, so

$$U(t\setminus\{j\}) = \sum_{1\le i\le n, i\ne j} (t\setminus\{v_j\}\boxplus i)$$

and hence

$$UD(t) = \sum_{1\le j\le k}\sum_{1\le i\le n, i\ne j}(t\setminus\{v_j\})\boxplus i.$$

Note that provided $i\ne j$, then the addition of a leaf at vertex v_i followed by the deletion of vertex v_j results in the same tree as the deletion of v_j followed by the addition of a leaf at vertex v_i. Thus $(t\boxplus i)\setminus\{v_j\} = (t\setminus\{v_j\})\boxplus i$, from which we obtain

$$(DU - UD)(t) = nt = (\rho(t)+1)t.$$

Although the commutator of U and D is not a scalar operator (so the theory developed in Chapter 2 will not apply to the poset $\mathcal{T}$), it is diagonal, and its diagonal values are constant on each rank. This chapter describes the theory used to study posets on which operators of this type can be defined.

3.1.2 Generalized Differential Posets

The discussion in Section 3.1.1 suggests that we should extend the theory of differential posets to accommodate posets with weights u and d on cover relations. We also wish to accommodate situations in which the commutator of U and D is not a scalar operator. Specifically, if A is an operator, and we use A_n to denote the restriction of A to $\mathbb{K}^{\mathcal{P}_n}$, then we would like to allow $(DU - UD)_n$ to depend on the rank n. (Note that for operators A and B, the notation AB_n may be used without ambiguity, since $(AB)_n$ and $A(B_n)$ have the same meaning.) Here is a definition which takes these considerations into account.

Definition 3.1. *Let $\mathbb{K}$ be a field of characteristic 0. A* **generalized differential poset** *is a collection $(\mathcal{P}, u, d, \{f_n\}_{n\in\mathbb{Z}})$, where*

1. *$\mathcal{P}$ is a locally finite ranked poset with finitely many elements of each rank,*
2. *$u, d : \mathcal{P} \times \mathcal{P} \to \mathbb{K}$ are functions such that $u(x, y) \neq 0$ or $d(x, y) \neq 0$ only if $x \lessdot y$,*
3. *and $\{f_n\}_{n\in\mathbb{Z}}$ is a sequence of polynomials such that*
$$DU_n = f_n(UD_n)$$
for all $n \in \mathbb{Z}$, where U and D are continuous linear operators on $\mathbb{K}^{\mathcal{P}}$ defined by
$$U(x) = \sum_{y\in\mathcal{P}} u(x, y)y$$
and
$$D(x) = \sum_{y\in\mathcal{P}} d(y, x)y.$$

In the special case where $(\mathcal{P}, u, d, \{f_n\}_{n\in\mathbb{Z}})$ is a generalized differential poset such that

$$u(x, y) = d(x, y) = \begin{cases} 1 & \text{if } x \lessdot y, \\ 0 & \text{otherwise,} \end{cases}$$

we say that the generalized differential poset is **unitary**. In this case, we often shorten the notation to $(\mathcal{P}, \{f_n\}_{n\in\mathbb{Z}})$.

The following observations about the definition of a generalized differential poset should be noted:

1. By taking $\mathcal{P}$ to be a unitary generalized differential poset with unique minimal element $\hat{0}$, and $f_n(z) = z + r$ for all n, this definition reduces to that of an r-differential poset.

2. Although in most combinatorial applications, the cover weights u and d are integer-valued, nothing in the theory requires this. Indeed, they can take any value in $\mathbb{K}$.

3. In typical applications, the polynomials f_n will be of the form $f_n(z) = q_n z + r_n$, though many aspects of the theory work for polynomials in general.

   Our ultimate goal is to compute the sum of weights of walks of shape W from x to y, where each walk is weighted by the product of the cover weights encountered as one traverses the walk. More specifically, if

$$C = x_0 x_1 \ldots x_k$$

   is a walk in which either $x_i \lessdot x_{i+1}$ or $x_{i+1} \lessdot x_i$ for $0 \leq i \leq k-1$, then define the weight of C to be

$$\omega(C) = \prod_{0 \leq i \leq k-1} \omega(x_i, x_{i+1})$$

   where

$$\omega(x,y) = \begin{cases} u(x,y) & \text{if } x \lessdot y, \\ d(y,x) & \text{if } y \lessdot x. \end{cases}$$

   Given a word W, and $x, y \in \mathcal{P}$, let $\mathfrak{C}(x \xrightarrow{W} y)$ denote the set of walks on cover relations of shape W, starting at x and ending at y. Let

$$e(x \xrightarrow{W} y) = \sum_{C \in \mathfrak{C}(x \xrightarrow{W} y)} \omega(C),$$

   the sum of weights of all such walks. It is clear, by an argument similar to the proof of Lemma 2.8, that

$$e(x \xrightarrow{W} y) = \langle y, Wx \rangle.$$

   As a shorthand, for $x \leq y$ in $\mathcal{P}$, let

$$e(x \rightarrow y) = e(x \xrightarrow{U^{\rho(y)-\rho(x)}} y)$$

   denote the number of saturated (x,y)-chains, weighted using the "upward" cover weights.

3.2 Generalized Differential Posets With Linear Commutation Relations

In this section, we look at some examples of generalized differential posets for which each f_n is a linear polynomial. We have already seen one such example — the poset of rooted trees.

3.2.1 The Induced Subgraph Poset

Let $\mathcal{V}$ be the set of all isomorphism classes of simple graphs. We can define an order on $\mathcal{V}$ by $G \leq H$ if and only if G is an induced subgraph of H. Note that, in this ordering, $G \lessdot H$ if and only if there exists a vertex v of H such that $H \setminus v$ is equal to G, which happens if and only if H can be obtained from G by adding a new vertex v adjacent to some subset of the vertices of G. This poset has a least element, the graph on no vertices, and it is ranked by $\rho(G) = |V(G)|$. Define the following weights on the cover relations: $d(G, H)$ is the number of vertices of H which, when deleted, result in G. $u(G, H)$ is the number of subsets $S \subset V(G)$ such that if a new vertex is added adjacent to every vertex in S, then the result is H. Thus, walks on this poset will correspond to the number of ways to change one unlabelled graph into another through a sequence of vertex additions or deletions. This poset is shown in Figure 3.2.

We now turn our attention to computing the commutator of U and D on this poset. Let $G \in \mathcal{V}$, with $\rho(G) = n$. Fix a labelling $\{1 \ldots, n\}$ of the vertices of G. For $S \subseteq \{1, \ldots, n\}$, let $G \boxplus S$ denote the graph obtained by adding the vertex $n+1$, adjacent to every vertex of S. Then

$$\begin{aligned}
DU(G) &= \sum_{S \subseteq \{1,\ldots,n\}} D(G \boxplus S) \\
&= \sum_{S \subseteq \{1,\ldots,n\}} \sum_{v \in \{1,\ldots,n+1\}} (G \boxplus S) \setminus \{v\} \\
&= \sum_{S \subseteq \{1,\ldots,n\}} \sum_{v \in \{1,\ldots,n\}} (G \boxplus S) \setminus \{v\} + 2^n G,
\end{aligned}$$

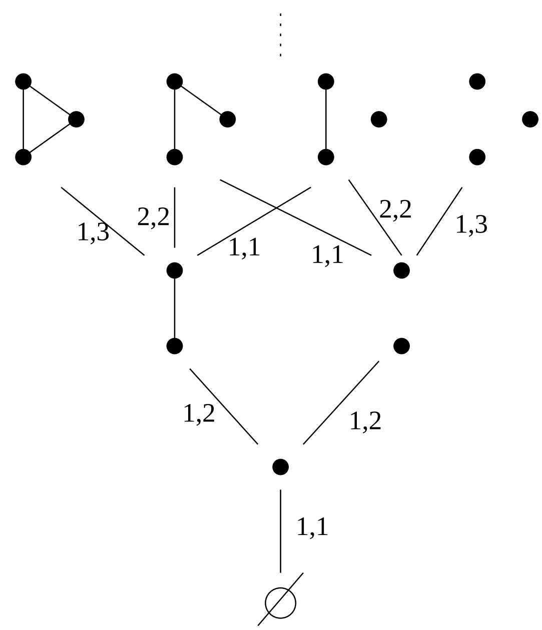


Figure 3.2: The poset $\mathcal{V}$ of induced subgraphs, with cover weights $u(G, H), d(G, H)$

and

$$
\begin{aligned}
UD(G) &= \sum_{v\in\{1,\ldots,n\}} U(G\setminus\{v\}) \\
&= \sum_{v\in\{1,\ldots,n\}} \sum_{S\subseteq\{1,\ldots,n\}\setminus\{v\}} (G\setminus\{v\})\boxplus S \\
&= \sum_{v\in\{1,\ldots,n\}} \sum_{S\subseteq\{1,\ldots,n\}\setminus\{v\}} (G\boxplus S)\setminus\{v\}.
\end{aligned}
$$

Note, however, that

$$
\sum_{S\subseteq\{1,\ldots,n\}\setminus\{v\}} (G\boxplus S)\setminus\{v\} = \sum_{S\subseteq\{1,\ldots,n\},v\in S} (G\boxplus S)\setminus\{v\},
$$

since for any S, the graph $(G\boxplus S)\setminus\{v\}$ is the same as $(G\boxplus(S\cup\{v\}))\setminus\{v\}$. So

$$
UD(G) = \frac{1}{2}\sum_{v\in\{1,\ldots,n\}} \sum_{S\subseteq\{1,\ldots,n\}} (G\boxplus S)\setminus\{v\},
$$

from which we obtain

$$
(DU - 2UD)(G) = 2^n G,
$$

so $\mathcal{V}$ is a generalized differential poset with $f_n(z) = 2z + 2^n$.

3.2.2 Combinatorial Conditions for Linear Commutation Relations

A particularly nice feature of linear commutation relations is that they often have a nice combinatorial interpretation, similar to the equivalence of the relation $DU - UD = rI$ to the combinatorial conditions of Theorem 2.3. Suppose that in a generalized differential poset, we have

$$
DU_n - q_n UD_n = r_n I. \tag{3.1}
$$

Let $x \in \mathcal{P}_n$. Then

$$
\begin{aligned}
DU_n(x) &= \sum_{y\in\mathcal{P}} u(x,y)D(y) \\
&= \sum_{z\in\mathcal{P}}\sum_{y\in\mathcal{P}} u(x,y)d(z,y)z,
\end{aligned}
$$

and

$$\begin{aligned} UD_n(x) &= \sum_{y \in \mathcal{P}} d(y,x)U(y) \\ &= \sum_{z \in \mathcal{P}} \sum_{y \in \mathcal{P}} u(y,z)d(y,x)z. \end{aligned}$$

So, equation (3.1) will hold if and only if the cover weights satisfy

$$\sum_{y \in \mathcal{P}} u(x,y)d(z,y) = q_n \sum_{y \in \mathcal{P}} u(y,z)d(y,x) \tag{3.2}$$

whenever $z \neq x$, and

$$\sum_{y \in \mathcal{P}} u(x,y)d(x,y) - q_n u(y,x)d(y,x) = r_n. \tag{3.3}$$

Thus, we have necessary and sufficient conditions on the cover weights for $\mathcal{P}$ to have linear commutation relations. The interpretation of these conditions becomes much clearer in the following section.

3.2.3 Sequentially Differential Posets

One specialization of generalized differential posets is to **sequentially differential posets**, also studied by Stanley [18]. A sequentially differential poset has a $\hat{0}$, is unitary, and has commutation relations given by $f_n(z) = z + r_n$, that is,

$$(DU - UD)_n = r_n I$$

for some sequence $\{r_n\}_{n \geq 0}$. Since $\mathcal{P}$ is unitary, then equation (3.2) becomes:

$$|\mathcal{U}_x \cap \mathcal{U}_z| = |\mathcal{D}_x \cap \mathcal{D}_z| \text{ if } x \neq z \tag{3.4}$$

which is the same condition as condition (2.4) for r-differential posets. Equation (3.3) becomes

$$|\mathcal{U}_x| = |\mathcal{D}_x| + r_n \tag{3.5}$$

whenever $x \in \mathcal{P}_n$. That is, if $\rho(x) = n$ then the number of elements which cover x exceeds the number of elements which x covers by r_n.

Here, we present a proof that finite Boolean lattices and projective geometries are sequentially differential posets. To prove this, it is useful to introduce the following class of posets, as defined by Stanley [15].

Definition 3.2. *A* **binomial poset** *is a locally finite poset with* $\hat{0}$*, containing finite chains of arbitrary length, satisfying:*

(a) All maximal chains in the interval $[x, y]$ *have the same length, denoted* $\ell(x, y)$*. If* $\ell(x, y) = n$*, the interval shall be referred to as an* n**-interval**.

(b) Any two n*-intervals contain the same number of maximal chains, denoted* $B(\mathcal{P}, n)$*, or simply* $B(n)$ *when the poset involved is clear from context.* $B(n)$ *is called the* **factorial function** *of the poset.*

Let $\mathcal{B}_{\text{fin}}(\mathbb{N})$ denote the set of finite subsets of $\mathbb{N}$, ordered by set inclusion. Note that any interval in $\mathcal{B}_{\text{fin}}(\mathbb{N})$ is isomorphic to $\mathcal{B}_n$ for some n, and that n is the length of the interval. Since any two n-intervals are isomorphic, then $\mathcal{B}_{\text{fin}}(\mathbb{N})$ is a binomial poset. Let $\mathcal{PG}_{\text{fin}}(q, \mathbb{N})$ denote the set of finite-dimensional subspaces of $\text{GF}(q)^{\mathbb{N}}$. In this poset, any interval is isomorphic to $\mathcal{PG}(q, n)$ for some n, with $n + 1$ being the length of the interval, from which it follows that $\mathcal{PG}_{\text{fin}}(q, \mathbb{N})$ is also a binomial poset.

Note that the interval $[x, y]$ in any binomial poset is ranked by $\rho(z) = \ell(x, z)$. Stanley motivates the use of the terms "binomial" and "factorial" by defining $\begin{bmatrix} n \\ i \end{bmatrix}$ to be the number of elements of rank i in an n-interval. If $[x, y]$ is an n-interval, note that for a fixed i, any maximal $[x, y]$-chain can be decomposed as a maximal $[x, z]$-chain and a maximal $[z, y]$-chain for some z of rank i. Thus,

$$B(i)B(n-i)\begin{bmatrix} n \\ i \end{bmatrix} = B(n)$$

or equivalently,

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{B(n)}{B(i)B(n-i)},$$

which has the same form as the classical binomial coefficients.

We can carry the analogy further by defining

$$A(n) := \begin{bmatrix} n \\ 1 \end{bmatrix} = \frac{B(n)}{B(n-1)}$$

to be the number of elements of rank 1 in an n-interval. (Note that $B(1) = 1$, since a 1-interval consists of a single element.) Then it is clear that

$$B(n) = \prod_{1 \le i \le n} A(i),$$

which has the same form as the classical factorial function. This formula may be used to easily compute $B(n)$. For example, in $\mathcal{B}_{\text{fin}}(\mathbb{N})$, an n-interval contains exactly n elements of rank one — namely, the singletons. Thus $A(n) = n$, so $B(n) = n!$. In $\mathcal{PG}_{\text{fin}}(q, \mathbb{N})$, an n- interval contains $\frac{q^n-1}{q-1}$ one-dimensional subspaces, so

$$B(n) = \prod_{1 \leq i \leq n} \frac{q^i - 1}{q - 1},$$

which is called the q-**factorial**.

The key result relating binomial posets to sequentially differential posets is as follows.

Lemma 3.3. *Let $\mathcal{P}$ be an n-interval in a binomial poset having factorial function $B(n)$. Then, for any $z \in \mathcal{P}$ such that $\rho(z) = i$,*

$$|\mathcal{U}_z| = |\mathcal{D}_z| + A(n-i) - A(i).$$

In particular, condition (3.5) is satisfied with $r_i = A(n-i) - A(i)$.

Proof: Let x, y be such that $\mathcal{P} = [x, y]$. Note that $\mathcal{U}_z$ is the number of elements of rank 1 in the interval $[z, y]$, so

$$|\mathcal{U}_z| = A(n-i).$$

Similarly, $\mathcal{D}_z$ is the number of elements of rank $i-1$ in the interval $[x, z]$, namely,

$$|\mathcal{D}_z| = \begin{bmatrix} i \\ i-1 \end{bmatrix} = \frac{B(i)}{B(i-1)} = A(i),$$

from which the result follows. □

As a corollary, we find that both $\mathcal{B}_n$ and $\mathcal{PG}(q, n)$ are sequentially differential posets.

Corollary 3.4. *$\mathcal{B}_n$ is a sequentially differential poset with $r_i = n - 2i$. $\mathcal{PG}(q, n)$ is a sequentially differential poset with $r_i = \frac{q^{n-i}-q^i}{q-1}$.*

Proof: Lemma 3.3 verifies condition (3.5) and gives the values for r_i. Condition (3.4) follows from the fact that both $\mathcal{B}_n$ and $\mathcal{PG}(q, n)$ are modular lattices.

□

3.3 Equivalence of Generalized Differential Posets and Oriented Graded Graphs

The definition of a generalized differential poset in Section 3.1 is (for combinatorial purposes) equivalent to a structure introduced by Fomin [3]. Fomin uses the language of graphs instead of that of partially ordered sets. A **graded graph** is a triple $G = (V, \rho, E)$ where V is a countable set of vertices, $\rho : V \to \mathbb{Z}$ is a grading, and E is a multiset of directed edges such that if $(x, y) \in E$, then $\rho(y) = \rho(x) + 1$. Given a pair of graded graphs $G_1 = (V, \rho, E_1)$ and $G_2 = (V, \rho, E_2)$ with a common vertex set and grading, form the **oriented graded graph** $G = (V, \rho, E_1, E_2)$ by interpreting an edge $(x, y) \in E_1$ as being directed from x to y, and interpreting an edge $(x, y) \in E_2$ as being directed from y to x. (Note that E_1 edges point in direction of increasing rank, and E_2 edges point in direction of decreasing rank.)

Up and down operators may be defined on $\mathbb{K}^V$, the $\mathbb{K}$-vector space spanned by the vertices, by

$$U(x) = \sum_{y,(x,y)\in E_1} y = \sum_{y\in V} a_1(x, y)y$$

and

$$D(x) = \sum_{y,(y,x)\in E_2} y = \sum_{y\in V} a_2(y, x)y,$$

where $a_i(x, y)$ denotes the multiplicity of the edge (x, y) in E_i. Fomin considered oriented graded graphs in which these operators satisfied a sequence of commutation relations $DU_n = f_n(UD_n)$.

Fomin's graph-based definition and the poset based definition are algebraically equivalent — in fact, many of the results referred to in this chapter were first proven by Fomin in the context of oriented graded graphs. This notion may be formalized by constructing a one-to-one correspondence between generalized differential posets and oriented graded graphs, which preserves the definitions of U and D. Let $G = (V, \rho, E_1, E_2)$ be an oriented graded graph. Define a partial order $\mathcal{P}$ on the ground set V by specifying $x \lessdot y$ if and only if $(x, y) \in E_1$ or $(x, y) \in E_2$, and taking the order $\leq$ to be the reflexive and transitive closure of $\lessdot$. Note that if $x \lessdot y$ in the poset, since $(x, y) \in E_1$ or $(x, y) \in E_2$, then $\rho(y) = \rho(x) + 1$, so ρ is a grading of this poset as well. Specify weights on the cover relations by

$u(x,y) = a_1(x,y)$ and $d(x,y) = a_2(x,y)$. Note that if y does not cover x in P, then the multiplicity of (x,y) as an edge in E_1 is zero, so $u(x,y) = 0$, and by a similar argument, $d(x,y) = 0$. But in this definition it is clear that the operators U and D, when defined in terms of G, agree in $\mathbb{K}^{\mathcal{P}}$ with the corresponding operators U and D as defined on the poset $(\mathcal{P}, \leq)$, so it is clear that the operators on $\mathcal{P}$ will satisfy the same sequence of commutation relations $DU_n = f_n(UD_n)$. Note that the cover weights in this poset are all integer-valued.

Conversely, let $(\mathcal{P}, u, d, \{f_n\}_{n\in\mathbb{Z}})$ be a generalized differential poset where u and d are integer-valued. Construct two directed multigraph versions of the Hasse diagram of $\mathcal{P}$ as follows. Let $G_1 = (\mathcal{P}, E_1)$ be a directed multigraph with vertex set $\mathcal{P}$, where $(x,y) \in E_1$ if and only if $x \lessdot y$, with the edge (x,y) occurring with multiplicity $u(x,y)$. Similarly, let $G_2 = (\mathcal{P}, E_2)$ have vertex set $\mathcal{P}$ and $(x,y) \in E_2$ if and only if $x \lessdot y$, with (x,y) occurring with multiplicity $d(x,y)$. Note that if $(x,y) \in E_i$, then $x \lessdot y$, so $\rho(y) = \rho(x) + 1$, so ρ is also a grading of each graph G_i. Then the oriented graded graph $G = (\mathcal{P}, \rho, E_1, E_2)$ has edge multiplicities $a_1(x,y) = u(x,y)$ and $a_2(x,y) = d(x,y)$, so it is clear that the operators U and D, defined in terms of G, agree with the definition in terms of the poset $(\mathcal{P}, \leq)$.

It is important to note that the vector space $\mathbb{K}^{\mathcal{P}}$, the grading ρ and the operators U and D are invariant under this correspondence. Thus, any result of a purely algebraic nature applies equally well to generalized differential posets and oriented graded graphs. Depending on what combinatorial application we have in mind, either the language of graphs or the language of posets may be used. Throughout this thesis, the language of posets is used.

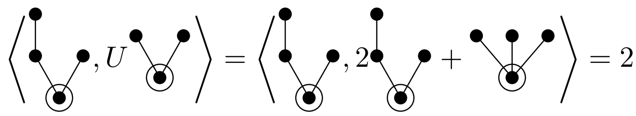
3.4 Spectral Analysis of UD_n and DU_n

3.4.1 The Adjoints of U and D and Some Combinatorial Comments

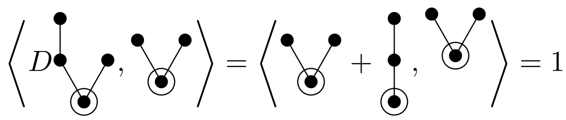
Recall that in Chapter 2 we introduced an inner product on $\mathbb{K}^{\mathcal{P}_n}$ given by

$$\langle x, y \rangle = \delta_{x,y}$$

and that, with respect to this inner product, the operator D is the adjoint of the operator U. It quickly becomes clear that this may not be the case for generalized differential posets. In the example of rooted trees,



but



However, under certain conditions, this inner product may be modified so that U and D are adjoint. The idea is to normalize the inner product by multiplying by some function $S : \mathcal{P} \to \mathbb{K} \setminus \{0\}$; that is, we shall use the inner product

$$\langle x, y \rangle_S = S(x)\delta_{x,y}.$$

Note that if $x \lessdot y$ in $\mathcal{P}$, then

$$\langle y, Ux \rangle_S = \left\langle y, \sum_{z \in \mathcal{P}} u(x,z)z \right\rangle_S = u(x,y)S(y)$$

and

$$\langle Dy, x \rangle_S = \left\langle \sum_{z \in \mathcal{P}} d(z,y)z, x \right\rangle_S = d(x,y)S(x).$$

Thus, in order for U and D to be adjoint, we require that

$$u(x,y)S(y) = d(x,y)S(x)$$

for all $x \lessdot y$ in $\mathcal{P}$.

In a r-differential poset, we have $u(x,y) = d(x,y)$ for all $x \lessdot y$, so the function given by $S(x) = 1$ for all $x \in \mathcal{P}$ is a trivial example of such a function (and, with this S, $\langle \cdot, \cdot \rangle_S$ is the inner product from Chapter 2). The poset $\mathcal{T}$ of rooted trees is an example of a generalized differential poset with a nontrivial function S, which comes from the following observation made by Hoffman. The definitions and results relating to the theory of group action may be found in many books on algebra, such as Artin [1]. Given a tree $t \in \mathcal{T}$, fix a labelling of the vertices of t, and let $\mathrm{Aut}(t)$ denote the group of automorphisms of t, that is, the group of permutations of the vertex set of t which fix the root and preserve the adjacency relationship. Let $t \lessdot t'$, and fix a labelling of t so that if the vertex w is added to t at the vertex v, then the result is t'. Let

$$\mathrm{Orb}(t, v) = \{u \in V(t) : \sigma(u) = v \text{ for some } \sigma \in \mathrm{Aut}(t)\}$$

denote the **orbit** of v in t, *i.e.* the set of vertices of t which can be mapped to v under some automorphism of t. It is clear that adding the vertex w to any vertex in $\mathrm{Orb}(t, v)$ results in a tree isomorphic to t'. Conversely, if u is a vertex of t to which adding w results in a tree isomorphic to t', then the restriction of this isomorphism to the vertices of t results in an automorphism of t which maps u to v. Thus, $u(t, t') = |\mathrm{Orb}(t, v)|$. A similar argument shows that $d(t, t') = |\mathrm{Orb}(t', w)|$.

Let

$$\mathrm{Stab}(v, t) = \{\sigma \in \mathrm{Aut}(t) : \sigma(v) = v\}$$

denote the **stabilizer** of v in $\mathrm{Aut}(t)$, i.e. the set of automorphisms that leave v fixed. By the Orbit-Stabilizer Theorem,

$$|\mathrm{Aut}(t)| = |\mathrm{Orb}(v, t)||\mathrm{Stab}(v, t)|,$$

from which we obtain

$$\frac{|\mathrm{Aut}(t)|}{u(t, t')} = |\mathrm{Stab}(v, t)|$$

and

$$\frac{|\mathrm{Aut}(t')|}{d(t, t')} = |\mathrm{Stab}(w, t')|.$$

Note that if $\sigma \in \mathrm{Stab}(v, t)$, then the permutation given by

$$\sigma'(u) = \begin{cases} w & \text{if } u = w, \\ \sigma(u) & \text{otherwise} \end{cases}$$

is an automorphism of t' fixing w, i.e. $\sigma' \in \mathrm{Stab}(w, t')$. Conversely, given any element $\sigma' \in \mathrm{Stab}(w, t')$, let σ denote its restriction to vertices of t. Note that σ is an automorphism of t and furthermore, that σ must fix v, since v is the unique parent of w. So $\sigma \in \mathrm{Stab}(v, t)$. Hence $|\mathrm{Stab}(v, t)| = |\mathrm{Stab}(w, t')|$, so

$$d(t, t')|\mathrm{Aut}(t)| = u(t, t')|\mathrm{Aut}(t')|,$$

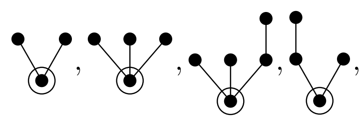
and thus the function $S(t) := |\mathrm{Aut}(t)|$ is such that U and D are adjoint with respect to $\langle \cdot, \cdot \rangle_S$.

Recall, as discussed in Section 2.2, that in the case of r-differential posets, the fact that U and D are mutually adjoint corresponds to the fact that we do not obtain different numbers of paths when traversing them in the reverse direction. This is not necessarily the case for generalized differential posets; indeed,

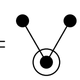
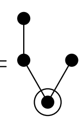
$$\langle y, Wx \rangle = \frac{1}{S(y)} \langle y, Wx \rangle_S = \frac{1}{S(y)} \langle W^* y, x \rangle_S = \frac{S(x)}{S(y)} \langle x, W^* y \rangle ,$$

i.e. $e(x \overset{W}{\to} y) = \frac{S(x)}{S(y)} e(y \overset{W^*}{\to} x)$. Thus, the function S determines the multiplicative factor by which weighted walk-enumeration results change when traversing paths backwards.

As an example, consider the DUU-walk in $\mathcal{T}$ given by



which has weight 6. (Note that this is the only walk of shape DUU with these endpoints.) Traversing this path in the opposite direction gives a weight of 3. With

$x =$  $, y =$ ,

noting that $S(x) = 2$ and $S(y) = 1$, we have verified in this case that $e(x \overset{W}{\to} y) = \frac{S(x)}{S(y)} e(y \overset{W^*}{\to} x)$.

It is interesting to note that if such a function S exists, then the operators UD and DU will be self-adjoint. Since the operators UD_n and DU_n are operators on the finite-dimensional space $\mathbb{K}^{\mathcal{P}_n}$, then they will be orthogonally diagonalizable. This suggests we search for a basis of eigenvectors for UD_n and DU_n. However, this observation is not entirely necessary since, as we shall see in the next section, the existence of an eigenbasis does not rely on the existence of the function S.

3.4.2 Eigenvalues and Eigenvectors of UD_n and DU_n

The eigenvalues and eigenvectors of UD_n and DU_n were computed by Stanley [17] for the case of r-differential posets, and his method was later generalized by Fomin in the context of oriented graded graphs. The technique relies on two key observations. The first is that the polynomial commutation relation $DU_n = f_n(UD_n)$ implies that if x is a λ-eigenvector of UD_n, then x is also a $f_n(\lambda)$-eigenvector of DU_n. The second observation follows from applying the operator U_n to both sides of the eigenvalue equation $DU_n x = \lambda x$.

$$\begin{aligned} \lambda U_n x &= UDU_n x \\ &= (UD)_{n+1}(U_n x), \end{aligned}$$

so $U_n x$ is an eigenvector of UD_{n+1} with eigenvalue λ. Furthermore, by the first observation, since $DU_{n+1} = f_{n+1}(UD_{n+1})$, then

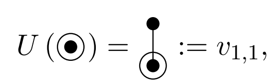
$$\begin{aligned} DU_{n+1}(U_n x) &= f_{n+1}(UD_{n+1})(U_n x) \\ &= f_{n+1}(\lambda) U_n x, \end{aligned}$$

so $U_n x$ is also an eigenvector of DU_{n+1} with eigenvalue $f_{n+1}(\lambda)$. We have proven the following.

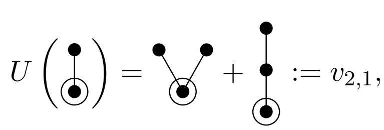
Lemma 3.5 (Fomin). *If x is an eigenvector of DU_n with eigenvalue λ, then $U_n x$ is an eigenvector of UD_{n+1} (with eigenvalue λ) and of DU_{n+1} (with eigenvalue $f_{n+1}(\lambda)$).*

Lemma 3.5 gives a recursive method for finding eigenvalues and eigenvectors of UD_n and DU_n. That is, once we have found one eigenvector of UD_k for some k, we can find some eigenvectors and eigenvalues for UD_n for $n \geq k$. (The question of whether we can find a basis of eigenvectors is not immediately clear, however.) In particular, when working with generalized differential posets with $\hat{0}$, we shall always have an eigenvector to initialize this procedure, since $\hat{0}$ is a 0-eigenvector of UD_0.

As an example of this method, consider the poset $\mathcal{T}$ from Section 3.1.1. Recall that the commutation relations on this poset are $f_n(z) = z + (n+1)$. The least element $v_{0,1} := \hat{0}$ is an eigenvector of UD_0 with eigenvalue 0, and so also an eigenvector of DU_0 with eigenvalue $f_0(0) = 1$. Applying U to the eigenvector $v_{0,1}$, we obtain

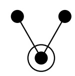
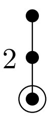


which is an eigenvector of UD_1 with eigenvalue 1. Thus, it is also an eigenvalue of DU_1 with eigenvalue $f_1(1) = 3$. Applying U to the vector $v_{1,1}$, we obtain



which is an eigenvector of UD_2 with eigenvalue 3, and hence also an eigenvalue of DU_2 with eigenvalue $f_2(3) = 6$.

We now run into a slight problem — $\mathcal{T}_2$ contains two elements, so we need two eigenvectors to span $\mathbb{K}^{\mathcal{T}_2}$. It is clear that any element in the kernel of D_2 will be a 0-eigenvector of UD_2, so let

$v_{2,2} :=$  $- 2$  ,

which is a 0-eigenvector of UD_2, and hence an eigenvector of DU_2 corresponding to the eigenvalue $f_2(0) = 3$. Continuing this procedure, we obtain the eigenvectors and eigenvalues shown in Table 3.1. As another example, some eigenvectors for Young's Lattice are given in Table 3.2.

With this example as a prototype, we now turn our attention to the problem of finding a general formula for the eigenvalues of UD_n, and their multiplicities. It should be clear from the above example that we will need to deal with repeated composition of various polynomials from the sequence $\{f_n\}_{n\in\mathbb{Z}}$. To handle this, we introduce the notation

$$f_{b\leftarrow a} = f_b \circ f_{b-1} \circ \ldots \circ f_{a+1} \circ f_a$$

for $b \geq a$. It is clear that $f_{b\leftarrow a}$ satisfies the identities

$$\begin{aligned} f_{a\leftarrow a} &= f_a, \\ f_b \circ f_{b-1\leftarrow a} &= f_{b\leftarrow a}, \\ f_{b\leftarrow a+1} \circ f_a &= f_{b\leftarrow a}. \end{aligned}$$

We will also adopt the notational convention that $f_{b\leftarrow a} = 1$ if $b < a$.

There is one slight problem with the method demonstrated in the preceding example, in that it does not keep track of the multiplicities of the eigenvalues. In particular, if S is a λ-eigenspace of DU_n, we know from Lemma 3.5 that $U(S)$ is a λ-eigenspace of UD_{n+1} and a $f_{n+1}(\lambda)$-eigenspace of DU_{n+1}. However, unless U is injective, it could be possible that $U(S)$ has a lower dimension than S. In order to keep track of the algebraic multiplicity of the eigenvalues, we use the following standard result regarding the relationship between the characteristic polynomials of AB and BA (see [20]). Denote by $\chi_T(x)$ the characteristic polynomial of T.

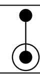
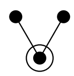
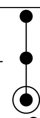
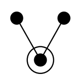
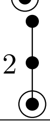
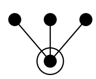
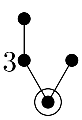
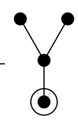
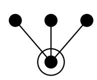
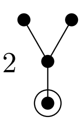
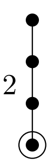
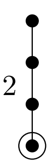
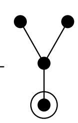
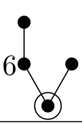
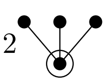
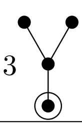
Rank	Eigenvectors	UD_n-eigenvalue	DU_n-eigenvalue
P_0		0	1
P_1		1	3
P_2	 $+$ 	3	6
	 $-\,2$ 	0	3
P_3	 $+\,3$  $+$  $+$ 	6	10
	 $-\,2$  $-\,2$ 	3	7
	2  $-$ 	0	4
	6  $-\,2$  $-\,3$ 	0	4

Table 3.1: Eigenvectors for the first 4 ranks of $\mathcal{T}$

Rank	Eigenvectors	UD_n-eigenvalues	DU_n-eigenvalues
$\mathcal{Y}_0$	$\hat{0}$	0	1
$\mathcal{Y}_1$	$\langle 1\rangle$	1	2
$\mathcal{Y}_2$	$\langle 11\rangle + \langle 2\rangle$	2	3
	$\langle 11\rangle - \langle 2\rangle$	0	1
$\mathcal{Y}_3$	$\langle 111\rangle + 2\langle 21\rangle + \langle 3\rangle$	3	4
	$\langle 111\rangle - \langle 3\rangle$	1	2
	$\langle 111\rangle - \langle 21\rangle + \langle 3\rangle$	0	1
$\mathcal{Y}_4$	$\langle 1111\rangle + 3\langle 211\rangle + 2\langle 22\rangle + 3\langle 31\rangle + \langle 4\rangle$	4	5
	$\langle 1111\rangle + \langle 211\rangle - \langle 31\rangle - \langle 4\rangle$	2	3
	$\langle 1111\rangle - \langle 22\rangle + \langle 4\rangle$	1	2
	$\langle 22\rangle - \langle 31\rangle + \langle 4\rangle$	0	1
	$\langle 1111\rangle - \langle 211\rangle + \langle 22\rangle$	0	1

Table 3.2: Eigenvectors and eigenvalues for the first five ranks of Young's Lattice

Lemma 3.6. *Let V and W be vector spaces, with $m = \dim V$, $n = \dim W$, where $n \geq m$. Let $A : V \to W$ and $B : W \to V$ be linear transformations. Then*

$$\chi_{AB}(x) = x^{n-m}\chi_{BA}(x).$$

In the context of generalized differential posets, we will be working with operators of the form $U : \mathbb{K}^{\mathcal{P}_n} \to \mathbb{K}^{\mathcal{P}_{n+1}}$ and $D : \mathbb{K}^{\mathcal{P}_n} \to \mathbb{K}^{\mathcal{P}_{n-1}}$, so in order to use this lemma, it is clear that the numbers

$$\Delta_n := |\mathcal{P}_n| - |\mathcal{P}_{n-1}|$$

will be important. This lemma tells us that if $\Delta_n \geq 0$, then

$$\chi_{UD_n}(x) = x^{\Delta_n}\chi_{DU_{n-1}}(x). \tag{3.6}$$

If $\Delta_n \leq 0$, then:

$$\chi_{DU_{n-1}}(x) = x^{-\Delta_n}\chi_{UD_n}(x)$$

so in fact, in this case, we can also write $\chi_{UD_n}(\lambda) = \lambda^{\Delta_n}\chi_{DU_{n-1}}(\lambda)$ without worrying about whether Δ_n is negative or positive.

We are now in a position to state and prove the general formula for the eigenvalues, which is Fomin's generalization of a result of Stanley for the case of r-differential posets.

Theorem 3.7 (Fomin). *Let $(\mathcal{P}, u, d, \{f_n\}_{n\geq 0})$ be a generalized differential poset, with a least element $\hat{0}$. Let*

$$\lambda_{n,k} = f_{n\leftarrow k}(0).$$

Then

$$\chi_{DU_n}(x) = \prod_{0\leq i\leq n} (x - \lambda_{n,i})^{\Delta_i},$$

and

$$\chi_{UD_n}(x) = x^{\Delta_n} \prod_{0\leq i\leq n} (x - \lambda_{n-1,i})^{\Delta_i}.$$

Proof: Proceed by induction on n. For the base case, $n = 0$, it is clear that 0 is the only eigenvalue of UD_0, and it has multiplicity $1 = \Delta_0$. Since $DU_0 = f_0(UD_0)$, then $f_0(0) = \lambda_{0,0}$ is the only eigenvalue of DU_0, and it has multiplicity 1.

For the inductive step, suppose $n \geq 1$ and that the theorem holds for smaller values than n. Thus,

$$\chi_{DU_{n-1}}(x) = \prod_{0\leq i\leq n-1} (x - \lambda_{n-1,i})^{\Delta_i},$$

so by equation (3.6),

$$\chi_{UD_n}(x) = x^{\Delta_n} \prod_{0\leq i\leq n-i} (x - \lambda_{n-1,i})^{\Delta_i}.$$

But since $UD_n = f_n(DU_n)$, then

$$\begin{aligned}\chi_{DU_n}(x) &= (x - f_n(0))^{\Delta_n} \prod_{0\leq i\leq n-i} (x - f_n(\lambda_{n-1,i}))^{\Delta_i} \\ &= \prod_{1\leq i\leq n} (x - \lambda_{n,i})^{\Delta_i}.\end{aligned}$$

Thus, by induction, the theorem is proven. $\square$

For special commutation relations, the polynomials $f_{b\leftarrow a}$ can be evaluated more explicitly. Assume the commutation relations are linear, *i.e.* $f_n(z) = q_n z + r_n$. Then, for $b \geq a$,

$$\begin{aligned}f_{b\leftarrow a}(z) &= f_b(f_{b-1}(\ldots(f_{a+1}(f_a(z))\ldots)) \\ &= q_b(q_{b-1}(\ldots(q_{a+1}(q_a z + r_a) + r_{a-1})\ldots) + r_{b-1}) + r_b \\ &= \prod_{a\leq i\leq b} q_i z + \sum_{a\leq i\leq b} r_i \prod_{i+1\leq j\leq b} q_j. \qquad (3.7)\end{aligned}$$

Specializing to the case when $q_n = 1$ for all $n \in \mathbb{Z}$,

$$f_{b\leftarrow a}(z) = z + \sum_{a \le i \le b} r_i, \tag{3.8}$$

and, when $r_n = r$ is constant, we obtain

$$f_{b\leftarrow a}(z) = z + (b - a + 1)r. \tag{3.9}$$

These polynomials will be of particular interest when evaluated at 0. For the most general linear case $f_n = q_n z + r_n$, we have

$$f_{b\leftarrow a}(0) = \sum_{a \le i \le b} r_i \prod_{i+1 \le j \le b} q_j. \tag{3.10}$$

If $q_n = 1$, we have

$$f_{b\leftarrow a}(0) = \sum_{a \le i \le b} r_i, \tag{3.11}$$

and for constant r, we have

$$f_{b\leftarrow a}(0) = (b - a + 1)r. \tag{3.12}$$

Knowledge of the eigenvalues alone can give us some information about the structure of the poset $\mathcal{P}$. In particular, if $\lambda_{n,k} \neq 0$ for all $0 \le k \le n$, then the operator DU_n will be invertible, implying that U is injective and D is surjective. The case in which U is injective is of particular interest, for then, applying U to an eigenspace will not reduce the dimension of the space. Also, injectivity of U implies that the rank sizes are weakly increasing, i.e. $|\mathcal{P}_j| \le |\mathcal{P}_{j+1}|$ for all $j \ge 0$.

In a number of examples, we find that U is injective. It is clear that for any rank of an r-differential poset, DU_n has only nonzero eigenvalues, as observed by Stanley [17]. In the poset of rooted trees, $r_n = n + 1 > 0$, so equation (3.11) implies that the eigenvalues of DU_n for $\mathcal{T}$ are all positive whenever $n \ge 1$, so the operator U is injective. Thus, given any distinct rooted trees on the same number of vertices, the multisets obtained by adding a leaf at each vertex are distinct.

The preceding observation about how the injectivity of U preserves eigenspace dimension suggests the following sufficient condition for diagonalizability of UD_n and DU_n.

Theorem 3.8 (Fomin). *Let $(\mathcal{P}, u, d, \{f_n\}_{n\geq 0})$ be a generalized differential poset with $\hat{0}$ such that, for all $0 \leq i \leq n$, and all $0 \leq j \leq i$, $\lambda_{i,j} \neq 0$. Let β_i be a linearly independent subset of* $\ker(D_i)$ *such that $|\beta_i| = |\mathcal{P}_i| - |\mathcal{P}_{i-1}|$ for $0 \leq i \leq n$. Then*

$$\bigcup_{0\leq i\leq n} U^{n-i}\beta_i$$

is a basis for $\mathbb{K}^{\mathcal{P}_n}$, and $U^{n-i}\beta_i$ consists of eigenvectors for UD_n with eigenvalues $\lambda_{n-1,n-i}$ and for DU_n with eigenvalues $\lambda_{n,n-i}$.

Proof: Proceed by induction on n. For the base case, $n = 0$, we have $\beta_0 = \hat{0}$, which clearly spans $\mathbb{K}^{\mathcal{P}_0}$. $\hat{0}$ is a 0-eigenvector for UD_0, and a $f_0(0) = \lambda_{0,0}$-eigenvector for DU_0. Suppose $n \geq 1$ and that a basis for $\mathbb{K}^{\mathcal{P}_{n-1}}$ exists, as described in the statement of the theorem. Since the eigenvalues $\lambda_{n-1,j} \neq 0$ for all $0 \leq j \leq n-1$, then U_{n-1} is injective. Thus, the set

$$\beta := U_{n-1}\left(\bigcup_{0\leq i\leq n-1} U^{n-1-i}\beta_i\right) = \bigcup_{0\leq i\leq n-1} U^{n-i}\beta_i$$

is linearly independent, and $|\beta| = |\mathcal{P}_{n-1}|$. Furthermore, by Lemma 3.5, $U^{n-i}(\beta_i) = U_{n-1}(U^{n-1-i}\beta_i)$ consists of vectors which are simultaneously eigenvectors of UD_n, with eigenvalue $\lambda_{n-1,n-i} \neq 0$, and eigenvectors of DU_n with eigenvalue $f_n(\lambda_{n-1,n-i}) = \lambda_{n,n-i}$.

By the rank-nullity formula,

$$\dim(\ker(D_n)) = |\mathcal{P}_n| - \operatorname{rank}(D_n) \geq |\mathcal{P}_n| - |\mathcal{P}_{n-1}|,$$

so there exists an independent set $\beta_n \subset \ker(D_n)$ of size $|\mathcal{P}_n| - |\mathcal{P}_{n-1}|$. It can be shown, as follows, that $\beta \cup \beta_n$ is linearly independent. Suppose there exist scalars a_v and a_w such that

$$\sum_{v\in\beta} a_v v = \sum_{w\in\beta_n} a_w w.$$

Since β consists of eigenvectors for UD_n with nonzero eigenvalues, apply UD_n to this equation to obtain

$$\sum_{v\in\beta} \lambda_v a_v v = 0,$$

where $\lambda_v \neq 0$ is the UD_n-eigenvalue for v. Thus, by independence of β, $a_v = 0$ for all $v \in \beta$. Thus

$$0 = \sum_{w \in \beta_n} a_w w,$$

so $a_w = 0$ for all $w \in \beta_n$, by independence of β_n. We see that $\beta \cup \beta_n$ is independent and has size $|\mathcal{P}_n|$, so it spans $\mathbb{K}^{\mathcal{P}_n}$, proving the theorem. $\square$

In the case of the Boolean lattice $\mathcal{B}_m$, if $n < \frac{m}{2}$, then $r_n = m - 2n > 0$, so the eigenvalues of DU_n are all positive. Thus, by Theorem 3.8, UD_n and DU_n will be diagonalizable for $0 \leq n < \frac{m}{2}$. For example, in $\mathcal{B}_4$, bases of eigenvectors can be found for $\mathcal{P}_0, \mathcal{P}_1$ and $\mathcal{P}_2$. Note, however, that

$$\lambda_{2,2} = f_2(0) = r_2 = 0,$$

so DU_2 is not invertible. Observe what happens when U is applied to the eigenvectors of UD_2.

$$\begin{aligned}
U(\{12\} + \{13\} + \{14\} + \{23\} + \{24\} + \{34\}) &= 3\left(\{123\} + \{124\} + \{134\} + \{234\}\right), \\
U(\{13\} + \{14\} - \{23\} - \{24\}) &= 2(\{134\} - \{234\}), \\
U(\{12\} + \{24\} - \{13\} - \{34\}) &= 2(\{124\} - \{134\}), \\
U(\{13\} + \{23\} - \{14\} - \{24\}) &= 2(\{123\} - \{124\}), \\
U(\{12\} - \{13\} - \{24\} + \{34\}) &= 0, \\
U(\{13\} - \{14\} - \{23\} + \{24\}) &= 0.
\end{aligned}$$

Though the last two computations resulted in the zero vector, the first four result in a set of vectors which span $\mathbb{K}^{\mathcal{P}_3}$. So, despite the fact that $\mathcal{B}_4$ does not satisfy the conditions of Theorem 3.8, we are still able to find a basis of UD_3-eigenvectors for $\mathcal{B}_4$.

The key to weakening the conditions of Theorem 3.8 lies in the observation that even though DU_2 is not invertible, UD_3 is. Thus, U_2 is surjective, so applying it to an eigenbasis for $\mathcal{P}_2$ results in a spanning set for $\mathbb{K}^{\mathcal{P}_3}$. This spanning set may be converted to a basis by selecting a linearly independent subset. Hence, we have the following stronger version of Theorem 3.8.

Theorem 3.9. *Let $(\mathcal{P}, u, d, \{f_n\}_{n \geq 0})$ be a generalized differential poset, and suppose that for all $0 \leq k \leq n$ at least one of UD_{n+1} or DU_n has only nonzero eigenvalues. Then there exists a basis*

$$\bigcup_{0 \leq i \leq n} \gamma_i$$

Rank	Eigenvectors	UD_n-eigenvalues	DU_n-eigenvalues
$\mathcal{P}_0$	$\emptyset$	0	4
$\mathcal{P}_1$	$\{1\}+\{2\}+\{3\}+\{4\}$	4	6
	$\{1\}-\{2\}$	0	2
	$\{2\}-\{3\}$	0	2
	$\{3\}-\{4\}$	0	2
$\mathcal{P}_2$	$\{12\}+\{13\}+\{14\}+\{23\}+\{24\}+\{34\}$	6	6
	$\{13\}+\{14\}-\{23\}-\{24\}$	2	2
	$\{12\}+\{24\}-\{13\}-\{34\}$	2	2
	$\{13\}+\{23\}-\{14\}-\{24\}$	2	2
	$\{12\}-\{13\}-\{24\}+\{34\}$	0	0
	$\{13\}-\{14\}-\{23\}+\{24\}$	0	0
$\mathcal{P}_3$	$\{123\}+\{124\}+\{134\}+\{234\}$	6	4
	$\{123\}-\{234\}$	2	0
	$\{124\}-\{134\}$	2	0
	$\{123\}-\{124\}$	2	0
$\mathcal{P}_4$	$\{1234\}$	4	0

Table 3.3: Eigenvectors for the Boolean lattice $\mathcal{B}_4$

for $\mathbb{K}^{\mathcal{P}_n}$, where $\gamma_i \subseteq U^{n-i}\beta_i$ is linearly independent. In particular, γ_i consists of eigenvectors which are simultaneously $\lambda_{n-1,n-i}$-eigenvectors for UD_n, and $\lambda_{n,n-i}$-eigenvectors for DU_n.

Proof: Use induction on n. For the base case, it is clear that $\hat{0}$ is an eigenvector of both UD_0 and DU_0, and $\{\hat{0}\}$ spans $\mathcal{P}_0$. For the inductive step, suppose we have a basis $\cup_{0\leq i\leq n-1}\gamma_i'$ for $\mathbb{K}^{\mathcal{P}_{n-1}}$ such that $\gamma_i' \subset U^{n-1-i}\beta_i$ is linearly independent, and consists of eigenvectors for UD_{n-1} and DU_{n-1}. If DU_n has only nonzero eigenvalues, then U is injective, so proceed as in the proof of Theorem 3.8.

If UD_{n+1} has only nonzero eigenvalues, then U is surjective. Thus,

$$U\left(\bigcup_{0\leq i\leq n-1}\gamma_i'\right) = \bigcup_{0\leq i\leq n-1} U(\gamma_i')$$

is a spanning set of $\mathbb{K}^{\mathcal{P}_n}$, so it contains a basis α. Let $\gamma_i = U(\gamma_i') \cap \alpha$, and note that γ_i is linearly independent. Let $\gamma_n = \emptyset$, and we have

$$\alpha = \bigcup_{0\leq i\leq n}\gamma_i$$

as a basis for $\mathbb{K}^{\mathcal{P}_n}$. Note that for $0 \leq i \leq n-i$,

$$\gamma_i \subseteq U(\gamma_i') \subseteq U(U^{n-1-i}\beta_i) = U^{n-i}\beta_i$$

and that $\gamma_n \subseteq \beta_n = \ker(D_n) = \emptyset$. In particular, γ_i' consists of eigenvectors of UD_n and DU_n, by Lemma 3.5. $\square$

3.5 Canonical Forms for Above-Words

Having computed the eigenvectors and eigenvalues for UD_n as described in Section 3.4.2, we now turn our attention to the problem of writing a monomial W in U and D as a polynomial g in UD_n. The eventual goal is to write elements of $\mathcal{P}$ in the bases of eigenvectors calculated in Section 3.4.2, which then reduces the problem of computing the action of W to the problem of evaluating g at the eigenvalues of UD_n. The results of the first part of this section apply to all generalized differential posets — we make no assumptions about the existence of a unique minimal element, and the only restriction on the sequence $\{f_n\}_{n\in\mathbb{Z}}$ is that each f_n is a polynomial.

As an example, consider the problem of writing the operator $UDUUDD_n$ as a polynomial in UD_n. We can apply the commutation relation $DU_n = f_n(DU_n)$ to obtain

$$UDUUDD_n = Uf_{n-1}(UD_{n-1})UDD_n.$$

Polynomials in UD_n satisfy a particularly nice property that allows us to continue with this computation. Note that if $g = \sum_{0\le i\le k} g_i z^i$ is any polynomial, then

$$\begin{aligned} g(UD_{n+1})U_n &= \sum_{0\le i\le k} g_i(UD_{n+1})^i U_n \\ &= \sum_{0\le i\le k} g_i U_n (DU_n)^i \\ &= U_n g(DU_n) \\ &= U_n g(f_n(UD_n)). \end{aligned}$$

By a similar argument, we also obtain $Dg(UD_n) = g(DU_{n-1})D_n = g(f_{n-1}(UD))D_n$. These observations (which are the content of Fomin's particularly useful Lemma 1.4.6) form the base case for the following generalization.

Lemma 3.10. *For any polynomial $g(z)$,*

$$\begin{aligned} g(UD)U_n^k &= U^k g(f_{n+k-1\leftarrow n}(UD_n)) \\ D^k g(UD_n) &= g(f_{n-1\leftarrow n-k}(UD_{n-k}))D_n^k. \end{aligned}$$

Proof: Use induction on k. The base case, $k = 1$, is done in the comments preceding the statement. For the inductive step, we have

$$\begin{aligned} g(UD)U_n^k &= (g(UD)U_{n+1}^{k-1})U_n \\ &= U^{k-1} g(f_{n+k-1\leftarrow n+1}(UD))U_n \\ &= U^k g(f_{n+k-1\leftarrow n+1}(f_n(UD_n)) \\ &= U^k g(f_{n+k-1\leftarrow n}(UD_n)). \end{aligned}$$

By induction, this formula holds for all $k \ge 1$. The other formula is proven in a similar manner. $\square$

Informally, this lemma allows us to "move U across a polynomial to the left" and to "move D across a polynomial to the right." By applying this to the computation preceding the lemma, we obtain

$$\begin{aligned}UDUUDD_n &= UUf_{n-1}(DU_{n-2})DD_n\\ &= UUf_{n-1}(f_{n-2}(UD_{n-2})DD_n\\ &= UUg(UD_{n-2})DD_n,\end{aligned}$$

where $g = f_{n-2} \circ f_{n-1}$. It is clear that we can go no further with this computation — the terminal string of D's is the problem, since we cannot apply the relation $DU_n = f_n(UD_n)$ to deal with it. This example illustrates that we may not be able to write a monomial as a polynomial in UD_n, unless it ends with a U. In fact, more generally, by writing

$$W = U^{a_m}DU^{a_{m-1}}D\dots DU^{a_1}DU^{a_0}$$

it soon becomes apparent that the condition we require is that $\sum_{0\le i\le k} a_i > k$ for all $0 \le k \le m$. A monomial satisfying this condition shall be called an **above-word**, since it corresponds to a walk that always stays above its starting point.

The strategy in dealing with an above-word is to focus on the subwords of the form DU^k. Note that, using Lemma 3.10,

$$\begin{aligned}DU_n^k &= DU_{n+k-1}U_n^{k-1}\\ &= f_{n+k-1}(UD)U_n^{k-1}\\ &= U^{k-1}f_{n+k-1}(f_{n+k-2\leftarrow n}(UD_n))\\ &= U^{k-1}f_{n+k-1\leftarrow n}(UD_n).\end{aligned}$$

A similar result holds for D^kU_n, giving the following.

Lemma 3.11 (Fomin).

$$\begin{aligned}DU_n^k &= U^{k-1}f_{n+k-1\leftarrow n}(UD_n)\\ D^kU_n &= f_{n\leftarrow n-k+1}(UD_{n-k+1})D_n^{k-1}.\end{aligned}$$

With this lemma in hand, we are now in a position to tackle the problem of deriving the canonical form for a general above-word, generalizing a result of Fomin.

Theorem 3.12. *Let $\mathcal{P}$ be a generalized differential poset with commutation relations $\{f_n\}_{n\in\mathbb{Z}}$, and let $W = U^{a_m}DU^{a_{m-1}}D\dots DU^{a_1}DU^{a_0}$ be an above-word. Let $b_i = \sum_{0\le j\le i} a_j - i$. Then there exists a polynomial $g_{W,n}(z)$ such* *that* $W_n = U^{b_m} g_{W,n}(UD_n)$. *In fact, such a polynomial is given by*

$$g_{W,n} = \prod_{0 \le i \le m-1} f_{n+b_i-1 \leftarrow n}.$$

Proof: Proceed by induction on m. When $m = 0$, we have $W = U^{a_0}$, so the theorem holds with $g_{W,n} = 1$. Suppose $m \ge 1$ and that the theorem holds for all smaller values of m. Applying Lemma 3.11,

$$\begin{aligned} W_n &= U^{a_m} D U^{a_{m-1}} D \dots D U^{a_1} D U_n^{a_0} \\ &= U^{a_m} D U^{a_{m-1}} D \dots D U^{a_1} U^{a_0 - 1} f_{n+a_0-1 \leftarrow n}(UD_n). \end{aligned}$$

Let $a'_0 = a_1 + a_0 - 1$ and $a'_i = a_{i+1}$ for $1 \le i \le m-1$. Consider the word

$$W' = U^{a'_{m-1}} D U^{a'_{m-2}} D \dots D U^{a'_0}.$$

Observe that for any $0 \le j \le m-1$,

$$\sum_{0 \le i \le j} a'_i = \sum_{0 \le i \le j+1} a_i - 1 > (j+1) - 1 = j,$$

so W' is an above-word. By induction,

$$W'_n = U^{b'_{m-1}} \prod_{0 \le i \le m-2} f_{n+b'_i-1 \leftarrow n}(UD_n)$$

where

$$b'_i = \sum_{0 \le j \le i} a'_i - i = \sum_{0 \le j \le i+1} a_i - (i+1) = b_{i+1},$$

so, by changing indices in the product expression for W',

$$\begin{aligned} W_n &= W' f_{n+b_0-1 \leftarrow n}(UD_n) \\ &= U^{b'_{m-1}} \left(\prod_{1 \le i \le m-1} f_{n+b'_{i-1}-1 \leftarrow n}(UD_n) \right) f_{n+b_0-1 \leftarrow n}(UD_n) \\ &= U^{b_m} \prod_{0 \le i \le m-1} f_{n+b_i-1 \leftarrow n}(UD_n). \end{aligned}$$

Taking $g_{W,n} = \prod_{0 \le i \le m-1} f_{n+b_i-1 \leftarrow n}$, the proof follows. $\square$

As an application of Theorem 3.12, consider the above-word $W = D^k U^{k+l}$. We have $a_0 = k + l$, and $a_i = 0$ for $1 \le i \le k$, and we obtain the following theorem of Fomin as a corollary.

Commutation Relation	Formula for $D^k U_{n-l}^{k+l}$
$f_n(z) = q_n z + r_n$	$U^l \prod_{1 \le i \le k} \Big(\prod_{n-l \le j \le n+k-i} q_j U D_{n-l}$ $+ \sum_{n-l \le j \le n+k-i} r_j \prod_{j+1 \le s \le n+k-i} q_s \Big)$
$f_n(z) = z + r_n$	$U^l \prod_{1 \le i \le k} \Big(U D_{n-l} + \sum_{n-l \le j \le n+k-i} r_j \Big)$
$f_n(z) = z + r$	$U^l \prod_{1 \le i \le k} (U D_{n-l} + (k + l - i + 1) r)$

Table 3.4: Formulae for $D^k U_{n-l}^{k+l}$

Theorem 3.13 (Fomin).

$$D^k U_{n-l}^{k+l} = U^l \prod_{1 \le i \le k} f_{n+k-i \leftarrow n-l}(U D_{n-l}).$$

A formula of particular interest is the case when $l = 0$.

Corollary 3.14 (Fomin).

$$D^k U_n^k = \prod_{1 \le i \le k} f_{n+k-i \leftarrow n}(U D_n).$$

With additional assumptions on the sequence of polynomials $\{f_n\}_{n \in \mathbb{Z}}$, the formula of Theorem 3.13 can be written in a more explicit form. Using the equations (3.7) to (3.9), we obtain the results shown in Table 3.4.

3.6 Enumerative Results

We have now developed enough algebraic machinery to make some enumerative statements about generalized differential posets. For the purposes of this section, we shall assume that the poset $\mathcal{P}$ has a least element $\hat{0}$. In this case, use the notation $e(x) = e(\hat{0} \to x)$ to denote the sum of weights of saturated $(\hat{0}, x)$-chains. It will be particularly easy to count walks starting at $\hat{0}$, since $D\hat{0} = 0$. Thus, for any polynomial $g(z)$, $g(UD_0)\hat{0} = g(0)\hat{0}$, since all terms of g vanish except for the constant term. This observation allows us to use Theorem 3.13 to derive a formula for the sum of weights of all $\hat{0} - x$ walks whose shape is a fixed above-word W as a multiple of $e(x)$.

Commutation Relation	$e(\hat{0} \xrightarrow{W} x)$
$f_n(z) = q_n z + r_n$	$e(x) \prod_{0 \le i \le m-1} \left(\sum_{0 \le i \le b_i - 1} r_i \prod_{i+1 \le j \le b_i - 1} q_j \right)$
$f_n(z) = z + r_n$	$e(x) \prod_{0 \le i \le m-1} \left(\sum_{0 \le i \le b_i - 1} r_i \right)$
$f_n(z) = z + r$	$e(x) r^m \prod_{0 \le i \le m-1} b_i$

Table 3.5: Formulae $e(\hat{0} \xrightarrow{W} x)$ for linear commutation relations and balanced above-word W

Corollary 3.15. *Let $W = U^{a_m} D U^{a_{m-1}} D \dots D U^{a_1} D U^{a_0}$ be any above-word. Let $\ell(W) = \sum_{0 \le i \le m} a_i - m$ be the displacement of W. Then, for every x such that $\rho(x) = \ell(W)$,*

$$e(\hat{0} \xrightarrow{W} x) = e(x) g_{W,0}(0) = e(x) \prod_{0 \le i \le m-1} f_{b_i - 1 \leftarrow 0}(0).$$

Proof: This is a straightforward calculation.

$$\begin{aligned}
e(\hat{0} \xrightarrow{W} x) &= \langle x, W_0 \hat{0} \rangle \\
&= \langle x, U^{\ell(W)} g_{W,0}(U D_0) \hat{0} \rangle \\
&= \langle x, U^{\ell(W)} \hat{0} \rangle g_{W,0}(0) \\
&= e(x) g_{W,0}(0) \\
&= e(x) \prod_{1 \le i \le m-1} f_{b_i - 1 \leftarrow 0}(0).
\end{aligned}$$

□

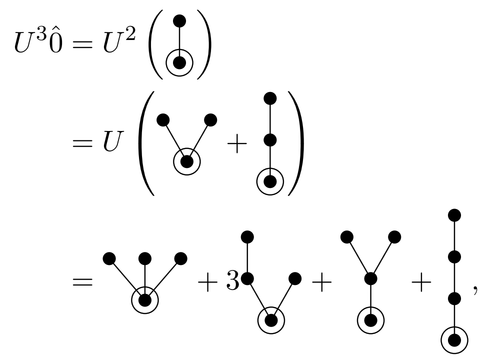
The specializations of this result to various linear commutation relations is shown in Table 3.5

It is interesting to note that among all $x \in \mathcal{P}_{\ell(W)}$, the number of W-walks starting at $\hat{0}$ and ending at x is a fixed multiple of $e(x)$. In particular, the probability of such a walk ending at any particular x depends only on $e(x)$, and not on the length or the shape of the walk, as illustrated in the following.

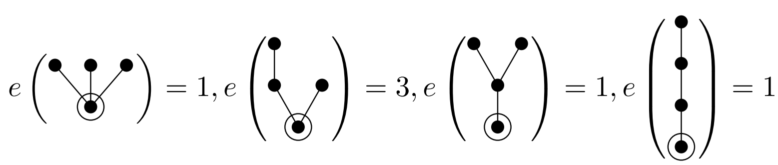
Example 3.16. A rooted tree with 3 non-root vertices is formed from a root via a sequence of additions or deletions of leaves. The sequence of operations is not known, though it is known that there are three more additions than deletions. Suppose we wish to calculate the probability of each possible outcome. We know that for each $t \in \mathcal{T}_3$,

$$e(\hat{0} \xrightarrow{W} t) = e(t)g_{W,0}(0)$$

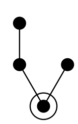
for some polynomial $g_{W,0}$ depending on W. We may compute $e(t)$ for each $t \in \mathcal{T}_3$ directly, as follows.



from which we obtain



There are a total of $6g_{W,0}(0)$ possible outcomes, with $e(t)g_{W,0}(0)$ outcomes resulting in the tree t. Hence the tree



occurs with probability 1/2 and the others occur with probability 1/6.

One interesting corollary is a formula for the number of walks which increase to a maximum, and then decrease.

Corollary 3.17 (Fomin). *For all $x \in \mathcal{P}$ with $\rho(x) = \ell(W)$,*

$$e(\hat{0} \xrightarrow{D^k U^{k+l}} x) = e(x) \prod_{1 \le i \le k} f_{k-i \leftarrow 0}(0).$$

Commutation Relation	$e(\hat{0} \overset{D^k U^{k+\ell}}{\to} x)$
$f_n(z) = q_n z + r_n$	$e(x) \prod_{1 \le i \le k} \left(\sum_{0 \le j \le k-i} r_j \prod_{j+1 \le s \le k-i} q_s \right)$
$f_n(z) = z + r_n$	$e(x) \prod_{1 \le i \le k} \left(\sum_{0 \le j \le k-i} r_j \right)$
$f_n(z) = z + r$	$e(x) r^k \prod_{1 \le i \le k} (k - i + 1)$

Table 3.6: Formulae for $e(\hat{0} \overset{D^k U^{k+\ell}}{\to} x)$ under linear commutation relations, where $\rho(x) = \ell$

The specializations of this result to various linear commutation relations is shown in Table 3.6.

A particularly interesting case of these formulae is when $x = \hat{0}$; that is, the case of closed walks of length $2k$. In this case $e(x) = 1$, so the walk lengths may be computed using only the sequence f_n. In particular, in the case of an r-differential poset, we obtain the familiar result

$$e(\hat{0} \overset{D^k U^k}{\to} \hat{0}) = r^k k!$$

which, in the case of Young's Lattice, is equation (1.1).

3.6.1 Enumerative Consequences of Spectral Analysis

In order to enumerate walks starting at points other than $\hat{0}$, we can use the spectral analysis carried out in Section 3.4. The general strategy for enumerating walks starting at x is to write x as a linear combination of eigenvectors of UD_n. Then, writing $W_n = U^\ell g_{W,n}(UD_n)$ as in Theorem 3.12, we can compute the action of W on x by evaluating $g_{W,n}$ at the eigenvalues of UD_n. This strategy is illustrated in the following.

Example 3.18. In Young's lattice, let σ denote the partition (21). Let $g_{\pi\setminus\sigma}$ denote the number of skew tableaux of shape $\pi \setminus \sigma$. Suppose we want to compute

$$\sum_{\pi \vdash 6, \sigma \le \pi} (g_{\pi\setminus\sigma})^2 = \langle \sigma, D^3 U^3 \sigma \rangle.$$

The eigenvectors of UD_n for Young's lattice are given in Table 3.2. Let $v_3 = (111) + 2(21) + (3)$ denote the 3-eigenvector, and let $v_0 = (111) - (21) + (3)$

denote the 0-eigenvector. Then $\sigma = \frac{1}{3}(v_3 - v_0)$. From Table 3.4,

$$g_{D^3U^3,3}(z) = (z+3)(z+2)(z+1),$$

so

$$\begin{aligned} D^3U^3\sigma &= \frac{1}{3}g_{D^3U^3,3}(UD_3)(v_3 - v_0) \\ &= \frac{1}{3}(g_{D^3U^3,3}(3)v_3 - g_{D^3U^3,3}(0)v_0) \\ &= 40v_3 - 2v_0 \\ &= 38(111) + 82(21) + 38(3). \end{aligned}$$

Then, for example,

$$\sum_{\pi \vdash 6}(g_{\pi\backslash\sigma})^2 = 82.$$

Computations may be simplified using the following result.

Theorem 3.19. *Let $g_{W,n}$ be as in Theorem 3.12. The eigenvalues of $g_{W,n}(UD_n)$ are given by*

$$\mu_{W,n,k} = \prod_{0\le i\le m-1} f_{n+b_i-1\leftarrow k}(0).$$

The $\mu_{W,n,k}$-eigenvectors of $g_{W,n}(UD_n)$ are the $\lambda_{n-1,k}$-eigenvectors of UD_n, for $0 \le k \le n-1$. Any 0-eigenvector of UD_n is a $\mu_{W,n,n}$-eigenvector of $g_{W,n}(UD_n)$.

Proof: For $0 \le k \le n-1$, if v is a $\lambda_{n-1,k}$-eigenvector of UD_n, then

$$\begin{aligned} g_{W,n}(UD_n)v &= g_{W,n}(\lambda_{n-1,k})v \\ &= \prod_{0\le i\le m-1} f_{n+b_i-1\leftarrow n}(f_{n-1\leftarrow k}(0))v \\ &= \prod_{0\le i\le m-1} f_{n+b_i-1\leftarrow k}(0)v, \end{aligned}$$

where we have used the computation of $\lambda_{n,k}$ from Theorem 3.7. By taking v to be a 0-eigenvector of UD_n, we can obtain $\mu_{W,n,n}$ by evaluating $g_{W,n}$ at 0. $\square$

The eigenvalues have the following form when applied to walks which increase to a maximum, and then decrease.

Commutation Relation	$\mu_{D^mU^m,n,k}$
$f_n(z) = q_n z + r_n$	$\prod_{1\le i\le m} \sum_{k\le j\le n+m-i} r_j \prod_{j+1\le s\le n+m-i} q_s$
$f_n(z) = z + r_n$	$\prod_{1\le i\le m} \sum_{k\le j\le n+m-i} r_j$
$f_n(z) = z + r$	$r^m \prod_{1\le i\le m}(n+m-i-k+1)$

Table 3.7: Values of $\mu_{D^mU^m,n,k}$ for linear commutation relations

Corollary 3.20.

$$\mu_{D^mU^m,n,k} = \prod_{1\le i\le m} f_{n+m-i\leftarrow k}(0).$$

The specializations of this formula under linear commutation relations are given in Table 3.7.

3.7 Further Results on Generalized Differential Posets

In [3], Fomin presents a large number of examples and applications of generalized differential posets beyond what is presented in this thesis. While this thesis focussed on the algebraic techniques used to enumerate walks in generalized differential posets, there are bijective results which generalize the Robinson-Schensted correspondence on Young's lattice, also due to Fomin [4].

Stanley's paper on sequentially differential posets [18] contains a generalization of Theorem 2.7 to the case of sequentially differential posets, namely

$$D_{n+1}\mathbf{P}_{n+1} = U_{n-1}\mathbf{P}_{n-1} + r_n\mathbf{P}_n.$$

He uses this to derive a formula for $\alpha(0 \to k)$, in terms of a sum indexed by involutions in $\mathfrak{S}_k$, which is a generalization of the formula for $\alpha(0 \to k)$ given at the end of Section 2.4.

Another note of interest regarding the spectral theory of Section 3.4 is that in the special case of Young's Lattice, it is possible to give a complete orthogonal set of eigenvectors for UD_n in terms of irreducible characters of $\mathfrak{S}_n$. This result is due to Stanley [17].

Chapter 4

Quotients of Generalized Differential Posets

4.1 Motivation

4.1.1 The Poset of Spanning Subgraphs

In this section, we examine another example of a generalized differential poset. It will not be immediately obvious that this poset is a generalized differential poset, motivating the development of further machinery which may be used to show it is a generalized differential poset.

Let $(\mathcal{E}_m, \leq)$ be the set of unlabelled spanning subgraphs of K_m, the complete graph on m vertices, ordered by $G \leq H$ if and only if G is a spanning subgraph of H. This poset is ranked, with $\rho(G)$ equal to the number of edges of G. Under this order, the cover relations are given by $G \lessdot H$ if and only if G is a spanning subgraph of H with exactly one less edge than H. Define weights on the cover relations as follows. Let $u(G, H)$ be the number of non-edges of G which, when added to G, result in H. Let $d(G, H)$ be the number of edges of H which, when deleted, result in G. The poset $\mathcal{E}_4$, with these cover weights, is shown in Figure 4.1. In this poset, the weight of a Hasse walk of shape W from G to H will represent the number of ways to transform G to H through a sequence of edge additions and deletions specified by W.

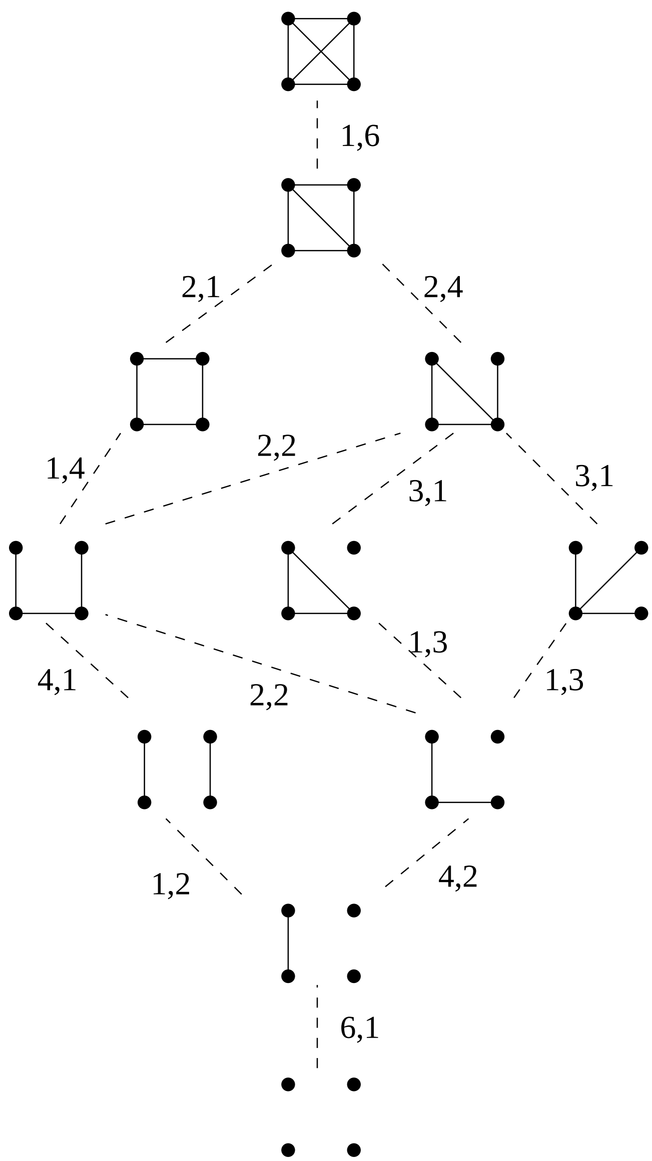
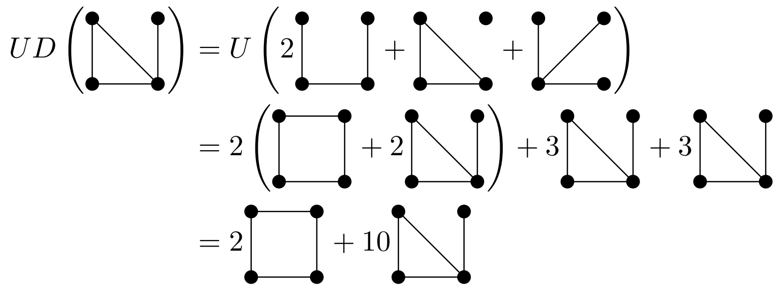
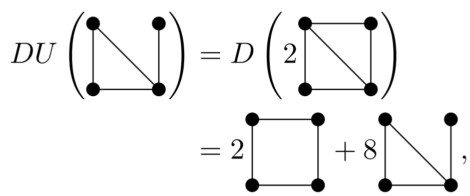


Figure 4.1: Poset of spanning subgraphs of K_4, with cover weights $u(x, y), d(x, y)$

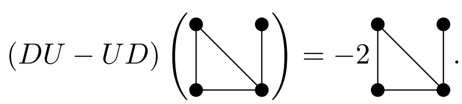
It is not immediately clear that $\mathcal{E}_n$ is a generalized differential poset. We can verify that $\mathcal{E}_4$ is a generalized differential poset by computing the commutator, for example,



and



from which we obtain



Evaluating the commutator at every other element of $\mathcal{E}_4$, we find that

$$(DU - UD)(G) = (6 - 2\rho(G))G,$$

so $DU_n = UD_n + (6 - 2n)I_n$. At this point it seems plausible to conjecture that in the poset $\mathcal{E}_m$, we have

$$DU_n = UD_n + \left(\binom{m}{2} - 2n\right) I_n. \tag{4.1}$$

Note that this commutator is the same as that for the Boolean lattice $\mathcal{B}_{\binom{m}{2}}$. Since a labelled graph on m vertices may be identified with a subset of $\binom{m}{2}$ possible edges, a reasonable strategy in proving equation (4.1) is to start with $\mathcal{B}_{\binom{m}{2}}$, and to reduce it modulo the graph-isomorphism equivalence relation in a way that preserves the commutator. The machinery for doing so is developed in this chapter.

4.1.2 Quotients of G.W. PECK Posets

The material developed in this chapter is similar to a result of Stanley dealing with quotients of G.W. PECK posets [16]. (G.W. PECK is usually abbreviated to PECK.) A finite, ranked poset $\mathcal{P}$, with maximum rank r, is said to be PECK over the field $\mathbb{K}$ if there exist operators $U_i : \mathbb{K}^{\mathcal{P}_i} \to \mathbb{K}^{\mathcal{P}_{i+1}}$ such that

1. for all $x \in \mathcal{P}_i$, $U_i(x) = \sum_{x \lessdot y} c_y y$ for some $c_y \in \mathbb{K}$, and
2. for all $0 \leq i \leq \lfloor \frac{r}{2} \rfloor$, the transformation

$$U_{r-i+1} \ldots U_i : \mathbb{K}^{\mathcal{P}_i} \to \mathbb{K}^{\mathcal{P}_{r-i}}$$

   is invertible.

An operator satisfying the first condition shall be referred to as a **raising operator**; a **lowering operator** is defined similarly. A PECK poset is said to be **unitary** if the constants c_y may all be taken to be 1. Stanley's main result from [16] is that the quotient of a unitary PECK poset (taken in a way that is similar to the method used in Section 4.2) is also a PECK poset.

A key result of Proctor [11] is that a poset $\mathcal{P}$ is PECK over $\mathbb{C}$ if and only if there are raising and lowering operators X and Y which span a representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, that is, if the commutator $H = XY - YX$ satisfies $H_j = (r - 2j)I_j$. In particular, every PECK poset is a generalized differential poset with $\mathbb{K} = \mathbb{C}$, $U = X$, $D = Y$, and $f_j(z) = z + (r - 2j)$.

It is generally known that it is possible to extend Stanley's result on PECK posets to generalized differential posets. In this chapter, we give a precise statement and proof of this "folklore theorem."

4.2 Quotients of Partially Ordered Sets

Given a ranked poset $(\mathcal{P}, \leq)$, an equivalence relation $\sim$ on $\mathcal{P}$ is said to be **rank-compatible** if $x \sim y$ implies $\rho(x) = \rho(y)$. If $\sim$ is rank-compatible, we can define a new poset $\mathcal{P}/\sim$ whose ground set is the set of $\sim$-equivalence classes. We can define an order relation $\leq_\sim$ by specifying the cover relations, then taking $\leq_\sim$ to be the reflexive, transitive and symmetric closure of $\lessdot$. Let $A \lessdot B$ in $\mathcal{P}/\sim$ if and only if there exist $x \in A$ and $y \in B$ such that $x \lessdot y$. In the following, the equivalence class containing x is denoted by $\tilde{x}$. Note that since $\sim$ is rank-compatible, then $\mathcal{P}/\sim$ has a well-defined rank function given by $\rho(\tilde{x}) = \rho(x)$.

In the case where $(\mathcal{P}, u, d, \{f_n\}_{n\in\mathbb{Z}})$ is a generalized differential poset, our objective is to assign weights $\tilde{u}(\tilde{x}, \tilde{y})$ and $\tilde{d}(\tilde{x}, \tilde{y})$ to the cover relations of $\mathcal{P}/\sim$ so that the operators $\tilde{U}$ and $\tilde{D}$ on $\mathbb{K}^{\mathcal{P}/\sim}$ satisfy the same commutation relations as $\mathcal{P}$, i.e. $\tilde{D}\tilde{U}_n = f_n(\tilde{U}\tilde{D}_n)$. The strategy in assigning these cover weights is to embed the vector space $\mathbb{K}^{\mathcal{P}/\sim}$ into $\mathbb{K}^{\mathcal{P}}$, perform computations using the known transformation U, and then reverse the embedding to obtain weights for $\mathbb{K}^{\mathcal{P}/\sim}$. (This will be made more precise in the following discussion.) If one regards a generalized differential poset as an oriented graded graph in the way described in Section 3.3, then the techniques used in this section are techniques from the theory of equitable partitions in algebraic graph theory (see [6]).

Lemma 4.1. *Let T be an injective linear transformation, $T : \mathbb{K}^{\mathcal{P}/\sim} \to \mathbb{K}^{\mathcal{P}}$, such that $\rho(T(\tilde{x})) = \rho(x)$ for all $x \in \mathcal{P}$. Suppose also that $UT = T\tilde{U}$ and $DT = T\tilde{D}$. Then $\tilde{D}\tilde{U}_n = f_n(\tilde{U}\tilde{D}_n)$.*

Proof: For any $\tilde{x} \in \mathcal{P}/\sim$, with $\rho(x) = n$,

$$\begin{aligned} T(\tilde{D}\tilde{U}_n\tilde{x}) &= DU_n(T\tilde{x}) \\ &= f_n(UD_n)(T\tilde{x}) \\ &= Tf_n(\tilde{U}\tilde{D}_n)\tilde{x}. \end{aligned}$$

Note that the final equality uses the fact that f_n is a polynomial. By injectivity of T, $\tilde{D}\tilde{U}_n\tilde{x} = f_n(\tilde{U}\tilde{D}_n)\tilde{x}$ for all $\tilde{x} \in \mathcal{P}/\sim$ such that $\rho(x) = n$. Thus, $\tilde{D}\tilde{U}_n = f_n(\tilde{U}\tilde{D}_n)$. □

Given an injective, linear, rank-preserving $T : \mathbb{K}^{\mathcal{P}/\sim} \to \mathbb{K}^{\mathcal{P}}$, by imposing the conditions $UT = T\tilde{U}$ and $DT = T\tilde{D}$, we can solve for the cover weights $\tilde{u}$ and $\tilde{d}$ in terms of u and d. Different embeddings will lead to different cover weights which all lead to the same commutation relations.

For example, consider the embedding given by $T(\tilde{x}) = \sum_{z\sim x} z$. Then

$$\begin{aligned} UT(\tilde{x}) &= U\left(\sum_{z\sim x} z\right) \\ &= \sum_{z\sim x}\sum_{y\in\mathcal{P}} u(z,y)y \\ &= \sum_{y\in\mathcal{P}}\left(\sum_{z\sim x} u(z,y)\right) y, \end{aligned}$$

and

$$\begin{aligned} T(\tilde{U}\tilde{x}) &= T\left(\sum_{\tilde{z}\in\mathcal{P}/\sim} \tilde{u}(\tilde{x},\tilde{z})\tilde{z}\right) \\ &= \sum_{\tilde{z}\in\mathcal{P}/\sim} \tilde{u}(\tilde{x},\tilde{z})\sum_{y\sim z} y \\ &= \sum_{y\in\mathcal{P}} \tilde{u}(\tilde{x},\tilde{y})y. \end{aligned}$$

Observe that the condition $UT = T\tilde{U}$ is satisfied if and only if

$$\tilde{u}(\tilde{x},\tilde{y}) = \sum_{z\sim x} u(z,y)$$

for all $\tilde{x},\tilde{y} \in \mathcal{P}/\sim$. We can obtain a similar formula for $\tilde{d}$, proving the following.

Theorem 4.2. *Let $(\mathcal{P}, u, d, \{f_n\}_{n\in\mathbb{Z}})$ be a generalized differential poset, and $\sim$ a rank-compatible equivalence relation. Let*

$$\tilde{u}(\tilde{x},\tilde{y}) = \sum_{z\sim x} u(z,y)$$

and

$$\tilde{d}(\tilde{x},\tilde{y}) = \sum_{z\sim y} d(x,z),$$

provided these values are well-defined. Then $(\mathcal{P}/\sim, \tilde{u}, \tilde{d}, \{f_n\}_{n\in\mathbb{Z}})$ is a generalized differential poset.

We can find a different set of cover weights by defining a new embedding, which is normalized by the size of the equivalence class $\tilde{x}$. Let T denote the embedding defined by

$$T : \tilde{x} \mapsto \frac{1}{|\tilde{x}|} \sum_{z \sim x} z.$$

Then

$$\begin{aligned} UT(\tilde{x}) &= \sum_{z \sim x} \frac{1}{|\tilde{x}|} U(z) \\ &= \sum_{z \sim x} \frac{1}{|\tilde{x}|} \sum_{y \in \mathcal{P}} u(z,y) y \\ &= \sum_{y \in \mathcal{P}} \left(\frac{1}{|\tilde{x}|} \sum_{z \sim x} u(z,y) \right) y, \end{aligned}$$

and

$$\begin{aligned} T(\tilde{U}\tilde{x}) &= \sum_{\tilde{z} \in \mathcal{P}/\sim} \tilde{u}(\tilde{x}, \tilde{z}) T(\tilde{z}) \\ &= \sum_{\tilde{z} \in \mathcal{P}/\sim} \tilde{u}(\tilde{x}, \tilde{z}) \frac{1}{|\tilde{z}|} \sum_{y \sim z} y \\ &= \sum_{y \in \mathcal{P}} \frac{1}{|\tilde{y}|} \tilde{u}(\tilde{x}, \tilde{y}) y. \end{aligned}$$

Imposing the condition $UT = T\tilde{U}$, we obtain the formula

$$\tilde{u}(\tilde{x}, \tilde{y}) = \frac{|\tilde{y}|}{|\tilde{x}|} \sum_{z \sim x} u(z,y),$$

for all $\tilde{x}, \tilde{y} \in \mathcal{P}/\sim$. Applying a similar calculation to the operator D, we obtain the following.

Theorem 4.3. *Let $(\mathcal{P}, u, d, \{f_n\}_{n \in \mathbb{Z}})$ be a generalized differential poset, and $\sim$ a rank-compatible equivalence relation. Let*

$$\tilde{u}(\tilde{x}, \tilde{y}) = \frac{|\tilde{y}|}{|\tilde{x}|} \sum_{z \sim x} u(z,y),$$

and

$$\tilde{d}(\tilde{x}, \tilde{y}) = \frac{|\tilde{x}|}{|\tilde{y}|} \sum_{z \sim y} d(x,z),$$

provided these values are well-defined. Then $(\mathcal{P}/\sim, \tilde{u}, \tilde{d}, \{f_n\}_{n \in \mathbb{Z}})$ is a generalized differential poset.

4.3 Quotients With Respect to Group Action

We now turn our attention to the situation in which $\sim$ is the equivalence relation of orbits with respect to some group action. Specifically, given a unitary generalized differential poset $(\mathcal{P}, \{f_n\}_{n\geq 0})$ with $\hat{0}$, let $\mathfrak{G}$ be a group of automorphisms of $\mathcal{P}$. Note that this implies that that $\rho(\sigma x) = \rho(x)$ for all $\sigma \in \mathfrak{G}$ and $x \in \mathcal{P}$, and also that if $x \lessdot y$, then $\sigma x \lessdot \sigma y$ for all $\sigma \in \mathfrak{G}$.

Let $\sim$ be the equivalence relation given by $x \sim y$ if and only if $x = \sigma y$ for some $\sigma \in \mathfrak{G}$. Observe that $\sim$ is rank-compatible, and that $\tilde{x} = \mathrm{Orb}(x)$. A natural set of cover weights on $\mathcal{P}/\sim$ would be

$$\tilde{u}(\tilde{x}, \tilde{y}) = |\{z \in \tilde{y} : x \lessdot z\}|, \tag{4.2}$$

i.e. the number of elements of the class $\tilde{y}$ which cover a fixed element of the class $\tilde{x}$, and

$$\tilde{d}(\tilde{x}, \tilde{y}) = |\{z \in \tilde{x} : z \lessdot y\}|, \tag{4.3}$$

the number of elements of the class $\tilde{x}$ which are covered by a fixed element of the class $\tilde{y}$. Our objective is to prove that with these cover weights, $(\mathcal{P}/\sim, \tilde{u}, \tilde{d}, \{f_n\}_{n\geq 0})$ is a generalized differential poset.

Our first attempt is to use Theorem 4.2. According to this theorem, using the cover weights

$$\tilde{u}(\tilde{x}, \tilde{y}) = \sum_{z\sim x} u(z, y)$$

and

$$\tilde{d}(\tilde{x}, \tilde{y}) = \sum_{z\sim y} d(x, z)$$

will result in a generalized differential poset. However, since $u(x, y) = 1$ when $x \lessdot y$ and 0 otherwise (and similarly for d) then

$$\tilde{u}(\tilde{x}, \tilde{y}) = |\{z \in \tilde{x} : z \lessdot y\}|$$

and

$$\tilde{d}(\tilde{x}, \tilde{y}) = |\{z \in \tilde{y} : x \lessdot z\}|.$$

To check that these are well-defined, suppose $\sigma, \pi \in \mathfrak{G}$. Then

$$\begin{aligned}
\tilde{u}(\widetilde{\sigma x}, \widetilde{\pi y}) &= |\{z \in \widetilde{\sigma x} : z \lessdot \pi y\}| \\
&= |\{\pi^{-1} z \in \widetilde{\pi^{-1}\sigma x} : \pi^{-1} z \lessdot y\}| \\
&= |\{z' \in \tilde{x} : z' \lessdot y\}| \\
&= \tilde{u}(\tilde{x}, \tilde{y}),
\end{aligned}$$

and similarly for $\tilde{d}$. Unfortunately, these are not the natural cover weights (4.2) and (4.3) that we would like — indeed, the roles of $\tilde{u}$ and $\tilde{d}$ are interchanged relative to (4.2) and (4.3).

Our next attempt is to use Theorem 4.3. Using this theorem, the weights

$$\tilde{u}(\tilde{x},\tilde{y}) = \frac{|\tilde{y}|}{|\tilde{x}|}\sum_{z\sim x} u(z,y),$$

and

$$\tilde{d}(\tilde{x},\tilde{y}) = \frac{|\tilde{x}|}{|\tilde{y}|}\sum_{z\sim y} d(x,z)$$

result in a generalized differential poset. (They are well-defined by the same argument presented in the preceding paragraph.) Thus, in order to prove (4.2) and (4.3), it suffices to prove that

$$|\tilde{x}||\{z\in\tilde{y} : x \lessdot z\}| = |\tilde{y}||\{z\in\tilde{x} : z\lessdot y\}| \tag{4.4}$$

whenever $x\lessdot y$. We will do this by constructing a bijection between the sets $\tilde{x}\times\{z\in\tilde{y} : x\lessdot z\}$ and $\tilde{y}\times\{z\in\tilde{x} : z\lessdot y\}$.

For each $a\in\tilde{x}$, fix an element $\sigma_a\in\mathfrak{G}$ such that $\sigma_a x = a$. For each $b\in\tilde{y}$, fix an element $\mu_b\in\mathfrak{G}$ such that $\mu_b y = b$. Define the function α by

$$\begin{aligned}\alpha : \tilde{x}\times\{z\in\tilde{y} : x\lessdot z\} &\to \tilde{y}\times\{z\in\tilde{x} : z\lessdot y\}\\ (a,c) &\mapsto (\sigma_a c, \mu^{-1}_{\sigma_a c} a).\end{aligned}$$

To check that this is well-defined, note that $c\in\tilde{y}$, so $\sigma_a c\in\tilde{y}$. Since $a\in\tilde{x}$, then $\mu^{-1}_{\sigma_a c}a\in\tilde{x}$. Since the action of $\mathfrak{G}$ preserves $\lessdot$, then

$$a = \sigma_a x\lessdot\sigma_a c$$

and hence

$$\mu^{-1}_{\sigma_a c}a\lessdot\mu^{-1}_{\sigma_a c}(\sigma_a c) = y.$$

In a similar manner, define the function β by

$$\begin{aligned}\beta : \tilde{y}\times\{z\in\tilde{x} : z\lessdot y\} &\to \tilde{x}\times\{z\in\tilde{y} : x\lessdot z\}\\ (b,d) &\mapsto (\mu_b d, \sigma^{-1}_{\mu_b d} b).\end{aligned}$$

Note that for any $(a, c) \in \tilde{x} \times \{z \in \tilde{y} : x \lessdot z\}$,

$$\begin{aligned}
\beta\alpha(a, c) &= \beta(\sigma_a c, \mu_{\sigma_a c}^{-1} a) \\
&= (\mu_{\sigma_a c}\mu_{\sigma_a c}^{-1} a, \sigma_a^{-1}\sigma_a c) \\
&= (a, c).
\end{aligned}$$

Similarly, we find that $\alpha\beta(b, d) = (b, d)$, so $\beta = \alpha^{-1}$. Thus, we have proven equation (4.4), and also the following theorem.

Theorem 4.4. *Let $(\mathcal{P}, \{f_n\}_{n\geq 0})$ be a unitary generalized differential poset with $\hat{0}$, and let $\mathfrak{G}$ be group of automorphisms of $\mathcal{P}$. Let $\sim$ be the equivalence relation defined by the orbits of $\mathfrak{G}$. Then $(\mathcal{P}/\sim, \tilde{u}, \tilde{d}, \{f_n\}_{n\geq 0})$ is a generalized differential poset, where*

$$\tilde{u}(\tilde{x}, \tilde{y}) = |\{z \in \tilde{y} : x \lessdot z\}|$$

and

$$\tilde{d}(\tilde{x}, \tilde{y}) = |\{z \in \tilde{x} : z \lessdot y\}|.$$

Note that equation (4.4) implies that

$$|\tilde{x}|\tilde{u}(\tilde{x}, \tilde{y}) = |\tilde{y}|\tilde{d}(\tilde{x}, \tilde{y}).$$

Let $S(\tilde{x}) := |\mathrm{Stab}(\tilde{x})|$. Then, by the Orbit-Stabilizer Theorem,

$$\tilde{u}(\tilde{x}, \tilde{y})S(\tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y})S(\tilde{x}),$$

so, by the discussion in Section 3.4.1, the inner product on $\mathbb{K}^{\mathcal{P}/\sim}$ defined by

$$\langle \tilde{x}, \tilde{y} \rangle_S = S(\tilde{x})\delta_{\tilde{x},\tilde{y}}$$

is such that $\tilde{D}$ is the adjoint of $\tilde{U}$. In particular, $\tilde{U}\tilde{D}_n$ and $\tilde{D}\tilde{U}_n$ are self-adjoint, and hence orthogonally diagonalizable, with respect to $\langle \cdot, \cdot \rangle_S$.

4.3.1 Application to the Poset of Spanning Subgraphs

We now turn our attention to using Theorem 4.4 to prove that the poset $\mathcal{E}_m$ of spanning subgraphs of K_m is a generalized differential poset. Let

E_m denote the set of two-element subsets of $\{1, \ldots, m\}$. Regard a vertex-labelled graph as an element of $\mathcal{B}(E_m) \cong \mathcal{B}_{\binom{m}{2}}$, which is a unitary generalized differential poset with

$$f_n(z) = z + \left(\binom{m}{2} - 2n\right).$$

Let $\mathfrak{G}$ be the symmetric group $\mathfrak{S}_m$ on m symbols. For $\sigma \in \mathfrak{G}$, define an action on $\mathcal{B}(E_m)$ by

$$\sigma\{\{a_1, b_1\} \ldots \{a_n, b_n\}\} = \{\{\sigma a_1, \sigma b_1\}, \ldots \{\sigma a_n, \sigma b_n\}\}.$$

Note that if $G \in E_m$, then σG and G are isomorphic as graphs — in fact, $\mathrm{Orb}(G)$ is the set of all graphs isomorphic to G. So $\mathcal{E}_m \cong \mathcal{B}(E_m)/\mathfrak{G}$.

We must check that $\mathfrak{G}$ is a group of automorphisms of $\mathcal{B}(E_m)$. It is clear from the definition that the group action is rank-compatible. Suppose that

$$G := \{\{a_1, b_1\} \ldots \{a_n, b_n\}\} \lessdot H := \{\{a_1, b_1\} \ldots \{a_n, b_n\}, \{a_{n+1}, b_{n+1}\}\}.$$

Then

$$\sigma G = \{\{\sigma a_1, \sigma b_1\}, \ldots \{\sigma a_n, \sigma b_n\}\}$$

and

$$\sigma H = \{\{\sigma a_1, \sigma b_1\}, \ldots \{\sigma a_n, \sigma b_n\}, \{\sigma a_{n+1}, \sigma b_{n+1}\}\},$$

so $\sigma G \lessdot \sigma H$. Thus, σ is an automorphism of $\mathcal{B}(E_m)$. We have proven the following.

Theorem 4.5. *$\mathcal{E}_m$ is a generalized differential poset with $\tilde{u}$ and $\tilde{d}$ as in Theorem 4.4, with $f_n(z) = z + \left(\binom{m}{2} - 2n\right)$.*

An interesting application of the fact that $\mathcal{E}_m$ is a generalized differential poset is to apply Theorem 3.7 to compute the eigenvalues of DU_n, namely,

$$\lambda_{n,k} = f_{n \leftarrow k}(0) = \sum_{k \le i \le n} \left(\binom{m}{2} - 2i\right),$$

for $0 \le k \le n$. Note that if $n < \frac{m(m-1)}{4}$, then $\binom{m}{2} - 2i$ is positive for all $0 \le i \le n$. Thus $\lambda_{n,k} > 0$ for all $0 \le k \le n$. In particular, DU_n is invertible, and hence U_n is injective, for all $0 \le n < \frac{1}{2}\binom{m}{2}$. Since $\mathcal{E}_n$ is isomorphic to its converse, D_n is injective for all $\frac{1}{2}\binom{m}{2} < n \le \binom{m}{2}$. This gives a solution to Harary's edge reconstruction problem (Problem II in [7]) in the case of graphs which have more that half the maximum number of edges, namely

Theorem 4.6. *Let G be a graph on m vertices and n edges. Let G_i denote the graph obtained by deleting the i^{th} edge. If $n > \frac{1}{2}\binom{m}{2}$, then G may be recovered up to isomorphism from $\tilde{G_1}, \ldots \tilde{G_n}$.*

This was first proven by Lovasz [9], and another proof was later given by Stanley [16] by taking quotients of PECK poset $\mathcal{B}(E_m)$ with respect to group action to obtain $\mathcal{E}_m$. The proof presented above is very similar to Stanley's proof — in fact, it is simply using the facts that the PECK posets used in Stanley's proof are also generalized differential posets, and that both PECK posets and generalized differential posets admit the same quotient operation with respect to group action.

Appendix A

Partially Ordered Sets

An understanding of some basic definitions related to partially ordered sets is essential to an understanding of this thesis. The intention of this appendix is not to give a comprehensive survey of results about partial orders, but rather, to provide only the background needed to understand this thesis. A detailed exposition may be found in a variety of sources, such as Crawley and Dilworth's book on lattices [2], or Stanley's book on enumerative combinatorics [19].

Definition A.1. *A* **partially ordered set** *(poset) is a set $\mathcal{P}$ with a binary relation $\leq_{\mathcal{P}}$ which satisfies the following set of axioms.*

1. *$x \leq_{\mathcal{P}} x$ for all $x \in \mathcal{P}$.*
2. *$x \leq_{\mathcal{P}} y$ and $y \leq_{\mathcal{P}} x$ implies $x = y$, for all $x, y \in \mathcal{P}$.*
3. *If $x \leq_{\mathcal{P}} y$ and $y \leq_{\mathcal{P}} z$ then $x \leq_{\mathcal{P}} z$ for all $x, y, z \in \mathcal{P}$.*

In cases where it is clear which poset is being discussed, we often write $\leq$ instead of $\leq_{\mathcal{P}}$.

A subset $\mathcal{Q} \subseteq \mathcal{P}$ is called an **induced subposet** (or, often, simply a "subposet") of $\mathcal{P}$ if for all $x, y \in \mathcal{Q}$, $x \leq_{\mathcal{Q}} y$ if and only if $x \leq_{\mathcal{P}} y$. Two posets $\mathcal{P}$ and $\mathcal{Q}$ are said to be **isomorphic** if there exists a bijection $f : \mathcal{P} \to \mathcal{Q}$ such that $f(x) \leq_{\mathcal{Q}} f(y)$ if and only if $x \leq_{\mathcal{P}} y$.

We will often deal with posets in which there exists a unique element $\hat{0}$

satisfying $\hat{0} \leq x$ for all $x \in \mathcal{P}$. (Similarly, we use $\hat{1}$ to denote the unique element such that $x \leq \hat{1}$ for all $x \in \mathcal{P}$, if such an element exists.)

An example of a partially ordered set is the set $\mathcal{B}(X)$ of subsets of a set X, with the order given by $A \leq B$ if and only if $A \subseteq B$. This is called the **Boolean lattice** on X. If $X = \{1, \ldots, n\}$, then $\mathcal{B}(X)$ is often written as $\mathcal{B}_n$. $\mathcal{B}(X)$ has a unique least element, $\hat{0} = \emptyset$, and a unique greatest element, $\hat{1} = X$. As another example, the **projective geometry** $\mathcal{PG}(q, n)$ is the set of all subspaces of the vector space $\mathrm{GF}(q)^{n+1}$, with $A \leq B$ if and only if A is a subspace of B. Its least element is the subspace consisting only of the zero vector, and its greatest element is $\mathrm{GF}(q)^{n+1}$.

For $x, y \in \mathcal{P}$, we say that y **covers** x, written $x \lessdot y$, if $x < y$ and for all $z \in \mathcal{P}$ such that $x \leq z \leq y$, either $x = z$ or $y = z$. Intuitively, $x \lessdot y$ if y is "immediately above x" in the partial order. As an example, in $\mathcal{B}_n$, $A \lessdot B$ if and only if $B = A \cup \{i\}$ for some $i \notin A$.

The **interval** $[x, y]$ is defined to be the induced subposet

$$[x, y] := \{z : x \leq z \leq y\}.$$

$\mathcal{P}$ is said to be **locally finite** if every interval is finite. Locally finite posets have a useful graphical representation called the **Hasse diagram** of the poset. The Hasse diagram of $\mathcal{P}$ is the graph with vertex set $\mathcal{P}$ and adjacency relationship given by $x \sim y$ if and only if either $x \lessdot y$ or $y \lessdot x$. A Hasse diagram is typically drawn such that elements which are greater in the poset are closer to the top of the page. The Hasse diagram of $\mathcal{B}_3$ is shown in Figure A.1

An element $x \in \mathcal{P}$ is said to be **minimal** if $y \leq x$ implies $y = x$. It is said to be **maximal** if $x \leq y$ implies $y = x$. The set of minimal elements of $\mathcal{P}$ is denoted $\mathcal{P}_{\min}$ and the set of maximal elements is denoted $\mathcal{P}_{\max}$. A poset is said to be **graded** if there exists a function $\rho : \mathcal{P} \to \mathbb{N}$ such that if $x \lessdot y$ then $\rho(y) = \rho(x) + 1$. The poset $\mathcal{P}$ is said to be **ranked** if $\rho(x) = 0$ for every $x \in \mathcal{P}_{\min}$ and $\rho(x) = r$ for every $x \in \mathcal{P}_{\max}$, for some constant r. As an example, $\mathcal{B}_n$ is ranked with $\rho(A) = |A|$ and $r = n$. $\mathcal{PG}(q, n)$ is ranked with $\rho(A) = \dim(A)$.

In a graded poset, we can define the n^{th} **level set** of $\mathcal{P}$ to be the set

$$\mathcal{P}_n := \{x \in \mathcal{P} : \rho(x) = n\}.$$

The sizes of the level sets can be recorded using the **rank-generating se-**

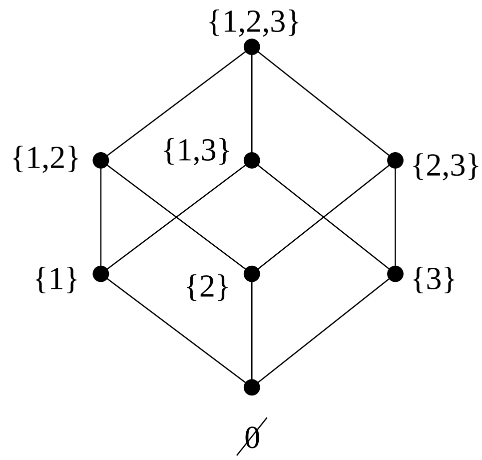


Figure A.1: Hasse diagram of the poset $\mathcal{B}_3$

ries, a formal power series given by

$$F(\mathcal{P}, q) := \sum_{x \in \mathcal{P}} q^{\rho(x)} = \sum_{n \geq 0} |\mathcal{P}_n| q^n.$$

As an example, note that in $\mathcal{B}_n$, the size of the k^{th} level set is $|\mathcal{B}_{n,k}| = \binom{n}{k}$, so

$$F(\mathcal{B}_n, q) = \sum_{0 \leq k \leq n} \binom{n}{k} q^k = (1+q)^n.$$

A.1 Lattices

For $x, y \in \mathcal{P}$, an element $a \in \mathcal{P}$ is said to be an **upper bound** for x and y if $x \leq a$ and $y \leq a$. It is said to be a **least upper bound** if for every $b \in \mathcal{P}$ such that $x \leq b$ and $y \leq b$, $a \leq b$. A greatest lower bound is defined similarly. A poset $\mathcal{L}$ is said to be a **lattice** if every pair of elements $x, y \in \mathcal{L}$ has a (necessarily unique) least upper bound, denoted by $x \vee y$, and a (necessarily unique) greatest lower bound, denoted by $x \wedge y$. The binary operations $\vee$ and $\wedge$ are referred to as **join** and **meet**, respectively. A subposet $\mathcal{M} \subseteq \mathcal{L}$ is a **sublattice** if for every pair of elements $x, y \in \mathcal{M}$, $x \vee y \in \mathcal{M}$ and $x \wedge y \in \mathcal{M}$, where $\vee$ and $\wedge$ are taken in $\mathcal{L}$. Examples of lattices include $\mathcal{B}(X)$, in which $\vee$ is set union and $\wedge$ is set intersection, and $\mathcal{PG}(q, n)$, in which $\vee$ is the direct sum of subspaces, and $\wedge$ is subspace intersection.

A lattice $\mathcal{L}$ is said to be **distributive** if it satisfies the additional axioms

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all $x, y, z \in \mathcal{L}$. These axioms are known as the distributive laws. It follows immediately from this definition that any sublattice of a distributive lattice is also distributive. Since set union and intersection satisfy these laws, then $\mathcal{B}(X)$ is distributive.

The following associates to any poset $\mathcal{P}$ a distributive lattice. Consider the distributive lattice $\mathcal{B}(\mathcal{P})$. An element $A \in \mathcal{B}(\mathcal{P})$ is said to be a **downward-closed set**, or simply a **down-set**, if $x \leq y$ and $y \in A$ implies $x \in A$. Let $J(\mathcal{P})$ denote the set of down-sets of $\mathcal{P}$. It is easy to check that the union of two down-sets is a down-set, as is the intersection of two down-sets, from which it follows that $J(\mathcal{P})$ is a sublattice of $\mathcal{B}(\mathcal{P})$. In particular, $J(\mathcal{P})$ is a distributive lattice.

Consider the poset $\mathbb{N}^2$, with order relation given by $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. Let $A \in \mathcal{B}(\mathbb{N}^2)$. A is a down-set if and only if for every $(a, b) \in A$, whenever $c \leq a$ and $d \leq b$ then $(c, d) \in A$. But this is exactly the condition required to ensure that A is the Ferrers diagram of a partition. Furthermore, the order on $J(\mathcal{P})$ (i.e. set containment) is exactly the order on Ferrers diagrams in Young's lattice $\mathcal{Y}$. So $\mathcal{Y} = J(\mathbb{N}^2)$, proving that $\mathcal{Y}$ is a distributive lattice.

A lattice $\mathcal{L}$ is said to be **modular** if and only if for all $a, b, c \in \mathcal{L}$,

$$(x \wedge y) \vee (x \wedge z) = x \wedge [y \vee (x \wedge z)].$$

It is clear from this definition that any distributive lattice is also modular. An equivalent formulation of this axiom is that a lattice is modular if and only if for all $x, y, z \in \mathcal{L}$ such that $x \leq y$,

$$x \vee (y \wedge z) = y \wedge (x \vee z).$$

Using this formulation along with some basic linear algebra, we can verify this condition for projective geometries. The following alternative characterization of modular lattices is useful in the study of differential posets which happen to be lattices.

Lemma A.2. *A lattice $\mathcal{L}$ is modular if and only if*

$$x, y \lessdot x \vee y \Leftrightarrow x \wedge y \lessdot x, y.$$

In particular, since Young's lattice is distributive, it is modular, so this condition holds.

Appendix B

The Method of Characteristics

The method of characteristics is a technique that can be used to reduce a special class of two-variable first-order partial differential equations to the easier problem of solving a system of ordinary differential equations. The material presented in this appendix may be found in many sources, such as [12]. The method of characteristics relies on a geometric interpretation of the solution, so technically, it only applies to real-valued functions of real variable. Though we wish to treat our functions as formal power series over the field $\mathbb{K}$, we can nevertheless use the geometric method as motivation for deriving a technique that uses only formal methods.

Let $f : \mathbb{R}^2 \to \mathbb{R}$, say $f = f(x, y)$. Let A, B and C be functions of f, x and y. Suppose we want to solve a partial differential equation of the form

$$A\frac{\partial f}{\partial x} + B\frac{\partial f}{\partial y} = C, \tag{B.1}$$

with initial condition $f(x, 0) = g(x)$. This equation may be re-written using the standard inner product on $\mathbb{R}^3$ as follows.

$$(A, B, C) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right) = 0.$$

Note that f determines a surface in $\mathbb{R}^3$ given by

$$S := \{(x, y, f(x, y)) : x, y \in \mathbb{R}\},$$

and that the normal vector to this surface is given by $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1)$. The vector (A, B, C) is always orthogonal to the normal, so it must be tangent to the surface S at each point $(x, y, f(x, y))$. In fact, for each point in $\mathbb{R}^3$, there is a curve through that point whose tangent is (A, B, C). These curves are called the **characteristic curves** of the partial differential equation (B.1).

We also know that the curve given by $(x, 0, g(x))$ lies in S. Thus, we may construct (at least part of) the surface S by finding the characteristic curves which pass through points of the form $(x, 0, g(x))$.

Let $\mathcal{C}$ be a characteristic curve. Parametrize $\mathcal{C}$ using the parameter τ so that

$$\mathcal{C} = \{(x(\tau), y(\tau), f(\tau)) : \tau \in \mathbb{R}\},$$

hence the tangent vector to $\mathcal{C}$ is given by

$$\left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{df}{d\tau}\right).$$

Since (A, B, C) is tangent to this curve, we obtain the system of ordinary differential equations

$$\begin{aligned} \frac{dx}{d\tau} &= A, \\ \frac{dy}{d\tau} &= B, \\ \frac{df}{d\tau} &= C, \end{aligned}$$

which will generally be easier to solve than the partial differential equation (B.1). The solution to this system of ordinary differential equations will involve some constants of integration; these may be determined using the initial condition $f(x, 0) = g(x)$. If we can write τ in terms of x and y, we will then have a solution of the form $f(x, y)$.

We are now in a position to give a description of the method of characteristics as it applies to formal power series, without recourse to geometric arguments. Suppose there are series f and τ which satisfy the ordinary differential equations $\frac{dx}{d\tau} = A$, $\frac{dy}{d\tau} = B$ and $\frac{df}{d\tau} = C$. Then, using the Chain Rule for formal power series,

$$C = \frac{df}{d\tau} = \frac{\partial f}{\partial x}\frac{dx}{d\tau} + \frac{\partial f}{\partial y}\frac{dy}{d\tau} = A\frac{\partial f}{\partial x} + B\frac{\partial f}{\partial y},$$

so f is a solution of the partial differential equation (B.1). If we can solve the three ordinary differential equations, we obtain an implicit description of a family of series f which satisfy (B.1). We then attempt to solve for f explicitly, using the initial condition $f(x, 0) = g(x)$ to pick out a unique solution. Note that this technique does not guarantee that one finds a solution; it merely provides sufficient conditions for a series f to be a solution.

A common simplification of this technique applies when $B = 1$. In this case, the ordinary differential equation $\frac{dy}{d\tau} = B$ has solution $y = \tau + y_0$ for some constant y_0. By renaming the parameter, we may assume without loss of generality that $y_0 = 0$, so $y = \tau$. So we have only two ordinary differential equations to solve, namely,

$$\frac{dx}{dy} = A \text{ and } \frac{df}{dy} = C.$$

Example B.1. Consider the partial differential equation

$$2f\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = -f,$$

with initial condition $f(x, 0) = 3x$. Applying the method of characteristics, we must solve the ordinary differential equations

$$\frac{dx}{dy} = 2f \text{ and } \frac{df}{dy} = -f.$$

The equation $\frac{df}{dy} = -f$ has solution

$$f = f_0 \exp(-y), \tag{B.2}$$

and the equation $\frac{dx}{dy} = 2f = 2f_0 \exp(-y)$ has solution

$$x = -2f_0 \exp(-y) + x_0. \tag{B.3}$$

The initial condition may be used to find the unknown f_0 and x_0. It is useful to write the initial curve, using an auxiliary parameter s, as

$$x = s, y = 0, f = 3s.$$

Along this curve, equation (B.2) becomes $3s = f_0$ and equation (B.3) becomes $s = -6s + x_0$. Substituting for x_0 in (B.3),

$$x = -6s \exp(-y) + 7s,$$

in other words,

$$s = \frac{x}{7 - 6\exp(-y)}.$$

Substituting $f_0 = 3s$ into equation (B.2), we obtain

$$f = \frac{3x\exp(-y)}{7 - 6\exp(-y)}$$

as our solution.

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